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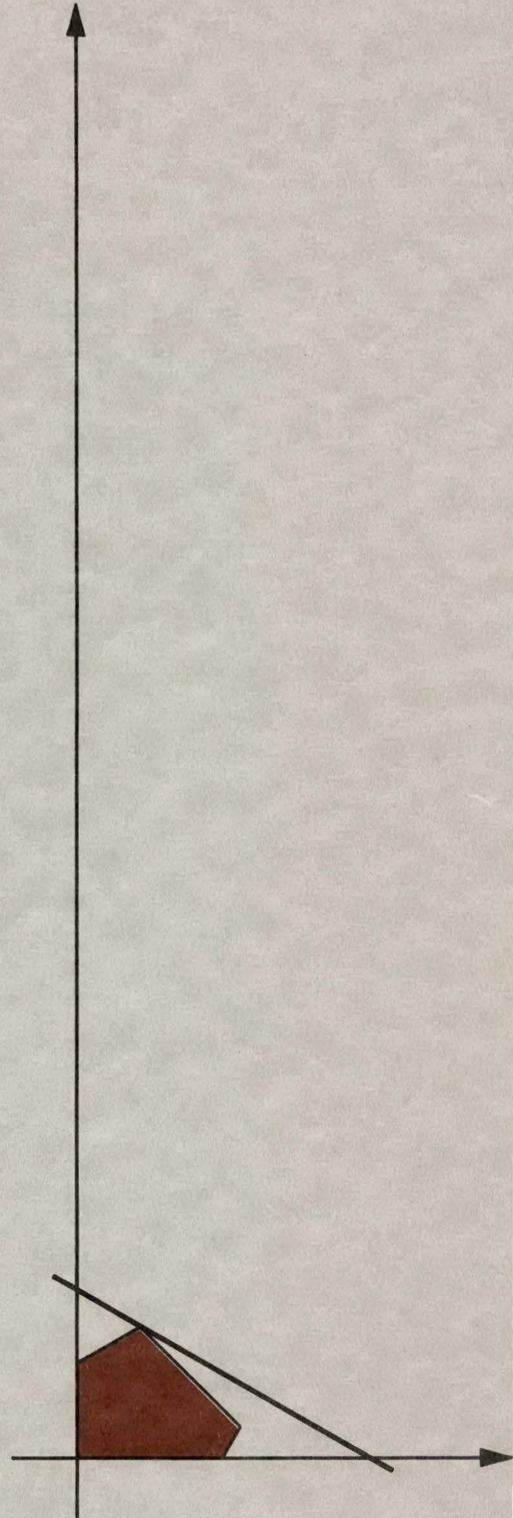
R. Chandrasekaran



OPERATIONS RESEARCH CENTER

COLLEGE OF ENGINEERING

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ABSTRACT

There are two broad categories of strategies used for fighting a wildland fire. They are (i) directly attacking the combustion with water, borate bombs or other extinguisher and (ii) indirectly constructing firebreaks which consist essentially in removing fuel along certain paths of some width which circumscribe the fire so as to stop further propagation of combustion. In this dissertation, we are concerned with determining the optimal firebreak paths for a fire of large conflagration.

First, we describe some fire spread models of interest. But we then restrict this analysis to the plane wave fire and treat the deterministic case where in the velocity of fire is known. Here we use variational calculus for determining the optimal path for given crew size and the number of construction groups into which the total crew splits and finally optimize over these variables. Then we show how an alternative approach using standard optimal control theory can be used to solve the variational problems and make a comparison between the two methods.

Finally, we consider the stochastic model where the velocity of fire is a random variable. Here we use a multistage dynamic programming model in which the dispatching of firebreak construction units is done periodically based upon the observed values of the velocity of fire spread at predetermined points of time. This analysis gives any planner the most important decision variables; viz, the initial crew size and the optimal number of review periods, each of duration T units of time, so that once a fire is reported, these can be used immediately.

CONTENTS

CHAPTER I

INTRODUCTION

1. Statement of the Problem

We are concerned here with optimal control of a forest fire, once it has become a large conflagration. There are two broad categories of strategies used in controlling a fire. They are: (i) directly attacking the combustion with water, borate bombs, or other extinguishers, and (ii) indirectly constructing firebreaks which consist essentially in removing fuel along certain paths of some width which circumscribe the fire so as to stop further spread. The first method [3, 4] is effective for early suppression of the development of a wildland fire, while the second is generally required for well-developed fires of large magnitude. In this dissertation, we are interested in the optimal construction of firebreaks.

Aside from the greater importance of constructing firebreaks since they apply to larger fires, the model of this method of fire fighting lends itself more easily to an evaluation and interpretation of the parameters involved, particularly when some of them are random variables, as compared with a model for direct suppression. Further, it is easier to describe free burning fire spread models for the study of optimal firebreaks.

2. Simplifying Assumptions

First we shall assume that the fuel for wildland fires is distributed uniformly on a flat plane. This assumption is obviously an idealization of the real situation, but it may be possible to develop, in an approximate way, projections of wildlands on a plane so that equal distances in

the direction of the climatological mean wind vector imply equal travel times for a free burning fire, allowing for the greater rate of travel of a fire up slopes as against the velocity of spread down slopes and also for variations in fuel distribution. Another way of interpreting this assumption in the real situation is to regard the idealized velocity of fire spread in the flat plane as an average velocity for the actual wildlands. In any event one cannot take account of every tree, bush, stream, rock, and slope.

Next, regarding fire spread models in the plane for free burning fires the fire front in still air is assumed to propagate radially with constant velocity as illustrated in Figure 1. If a mean wind vector is superimposed, the movement of the fire front is assumed to be modified by the addition of a fixed velocity vector for the center of the fire in the direction of the mean wind, resulting in a fire front spread as illustrated in Figure 2. Generally, the rate of propagation in still air is small relative to the wind effect and the contours of fire spread in the presence of wind will be long and narrow, suggesting an approximation in the form of a plane wave front moving with constant velocity in a channel of fixed width L as illustrated in Figure 3.

In this analysis we shall restrict ourselves to the plane wave fire spread model of Figure 3. The extension to the radially expanding fire when the climatological wind is negligible is rather straightforward and if we drop the assumption of approximating the fire front by a plane wave the extension is conceptually straightforward but computationally complex.

In the second chapter we consider the deterministic case wherein the velocity of fire spread is known. In the third chapter we compare the techniques of variational calculus and the standard optimal control theory techniques as applied to this problem.

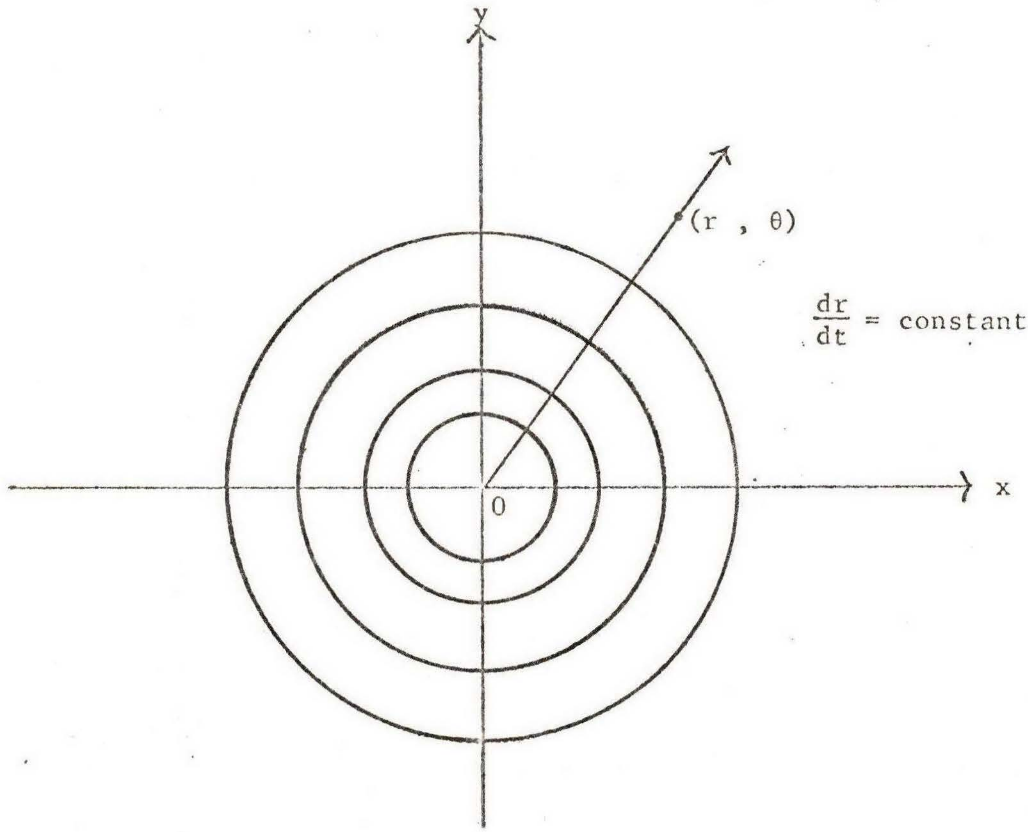


FIGURE 1. FIRE SPREAD IN STILL AIR

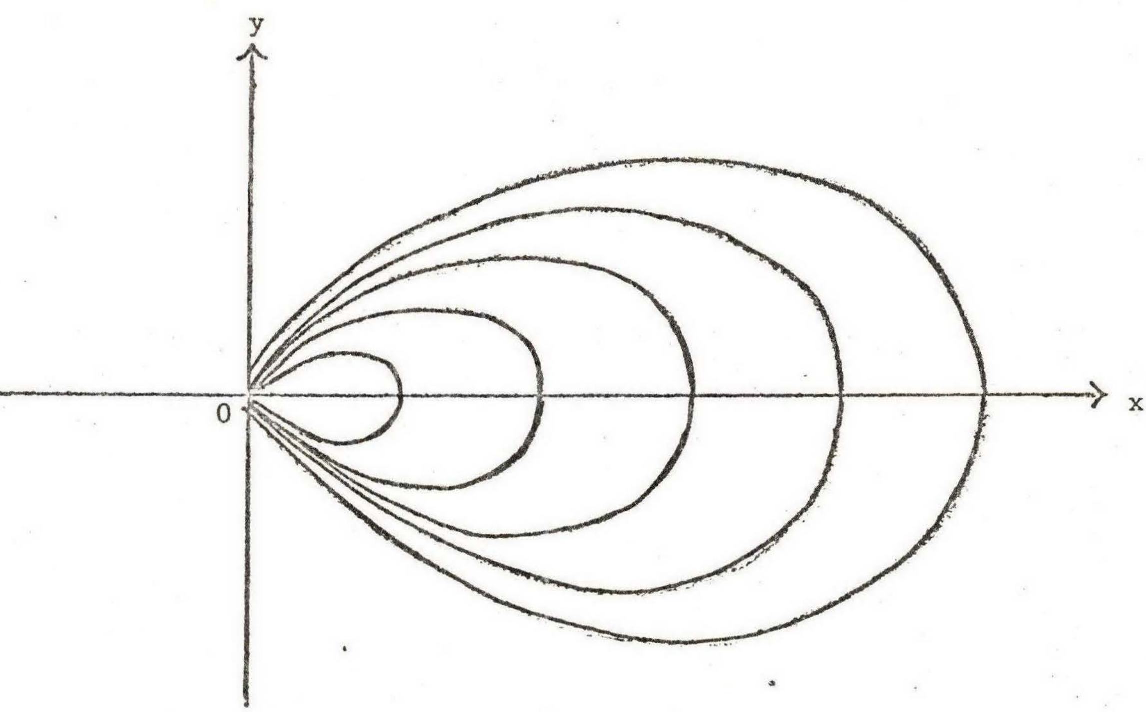


FIGURE 2. FIRE SPREAD MODIFIED BY WIND VECTOR

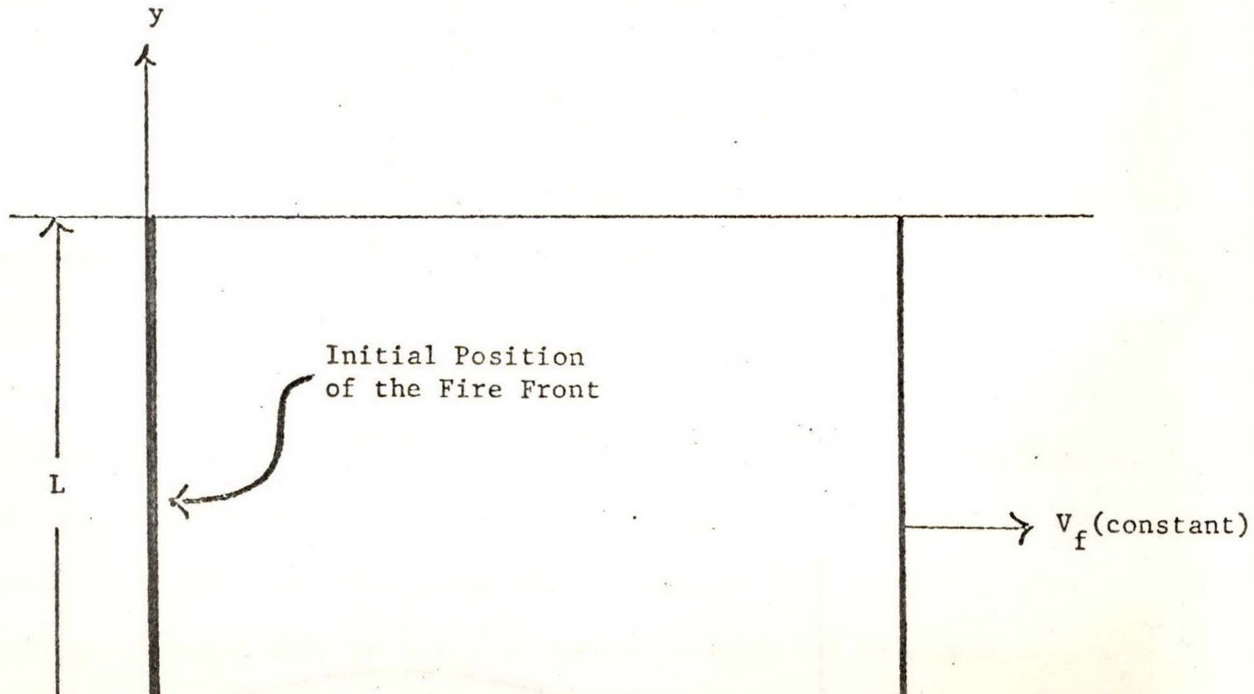


FIGURE 3. PLANE WAVE FIRE SPREAD

Then in the fourth chapter we extend the analysis to the case when the velocity of fire spread is a random variable. Here we use a periodic review policy for the optimal construction of firebreaks, using dynamic programming to determine the sequential alteration of the optimal firebreak paths for arbitrary sequences of realized values of the randomly varying wind. Then, we give a numerical algorithm for a computational solution. Finally we comment on the feasibility of computation and the importance of such an analysis for practical purposes. We show that, for planning purposes, such a model is useful and though the computation is large, it has to be done once and for all for a forest and just by knowing the cost parameters and the probability distribution functions which can be obtained by studying the forest for some time, we can determine the optimal number of periods p^* and the initial crew size N_1^*. These are the quantities that require the largest fraction of time for computation. Thus the analysis shows that such computations are feasible since they have to be carried out once and for all.

CHAPTER II

DETERMINISTIC MODEL

1. Introduction

In this chapter, we consider the fire spread model of Figure 3 with the assumption that the velocity of fire is known with certainty and is a constant over the width of the fire. First we shall describe the method of constructing firebreaks with an illustration of how we get a continuous firebreak for different numbers of construction groups. Then we state the cost structure used for the analysis and thereafter describe the procedure of getting the optimal firebreak for given values of crew size and number of construction groups. Finally we optimize on the crew size and the number of construction groups.

2. Method of Constructing Firebreaks

Men with hand tools and portable equipment are used to construct a firebreak. A total crew of N men is split into n equal groups and each of these groups constructs one nth of the firebreak. For n even or odd, the firebreak sections are laid out symmetrically in a continuous path as illustrated in Figures 4 and 5. Starting at points P, a section of the firebreak is constructed to a point Q. Thus the fire may be subdivided into n contiguous plane wave fires, and we need only study for given total crew size N and number of construction groups n the optimal firebreak path for a plane wave fire of width L/n. With these optimal paths, depending upon N and n, one may determine crew size and group size so as to minimize the resulting total costs. For this analysis it is assumed that the time required to distribute the fire fighters along a working line is negligible.

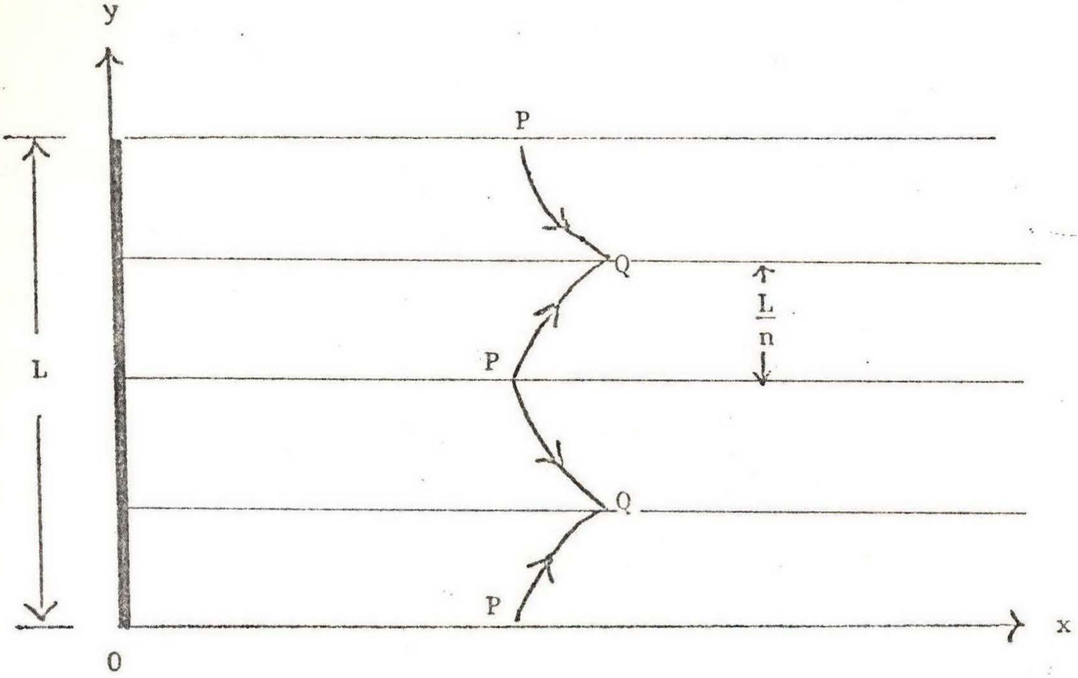


FIGURE 4. SECTIONALIZED FIREBREAK FOR n = 4

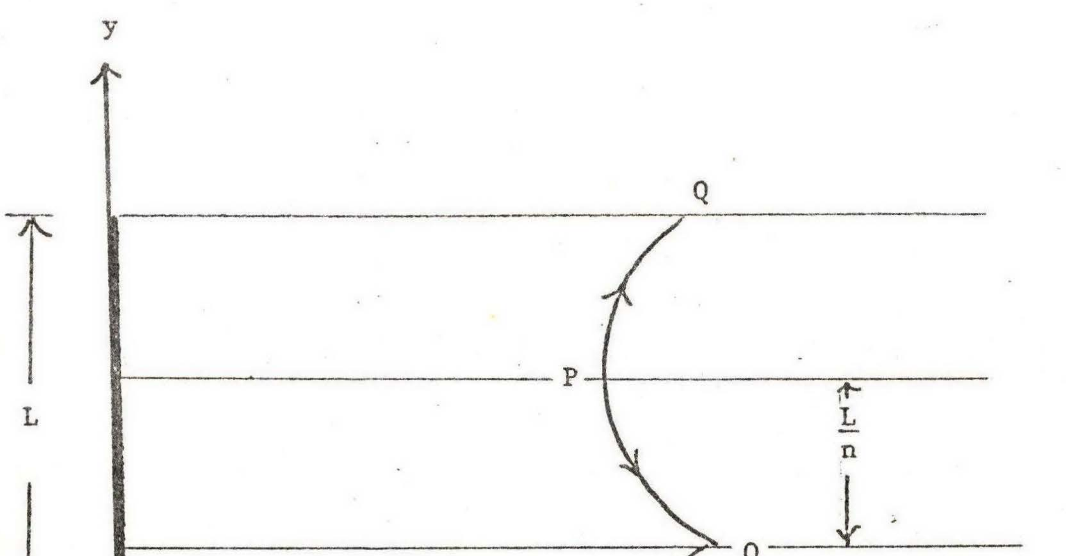


FIGURE 5. SECTIONALIZED FIREBREAK FOR n = 3

3. Cost Structure

The cost structure proposed by Parks and Jewell [5] is used. It includes the following four types of cost:

(i) A fixed cost associated with maintaining a fire fighting organization and setting it into action at the time of a fire, denoted by C_F (in dollars).

(ii) A cost proportional to the crew size N used, including items such as transportation, and other "one-shot" logistic support costs, denoted by $C_S \cdot N$ where C_S is dollars per man.

(iii) A cost proportional to the total number of man hours used in constructing firebreaks, denoted by $C_m \cdot N \cdot T_c$ where C_m is dollars per man hour and T_c is time of control.

(iv) A cost measuring the fire damage which is proportional to the area burnt, denoted by $C_B \cdot A$ where C_B is dollars per acre and A is the total area burnt.

In these terms the total cost K for any fire attacked by a crew of size N is given by

$$K = C_F + C_S \cdot N + C_m \cdot N \cdot T_c + C_B \cdot A \quad . \tag{1}$$

Although not explicitly indicated in (1), the total cost depends upon the path chosen for the firebreak and the number of construction groups n, via the quantities T_c and A. In order to see this we need to consider details on extremal firebreak paths for a plane wave fire of width L/n.

4. Extremal Firebreak Paths (Given N and n)

Define the firebreak path by a real, nonnegative single-valued function $X_m = f(y)$, denoting the distance of the firebreak from the

initial position of the fire front for all values of y in the interval $[0, L/n]$. In particular $f(0) = X_o \geqq 0$ (see Figure 6).

For some nominal width W of the firebreak, let Δ denote the firebreak area constructed per man per unit time. Then the ratio Δ/W determines the velocity V_m of firebreak construction per man used.

The position of the fire front at any time t is given by

$$X_f(t) = V_f \cdot t \tag{2}$$

where V_f denotes the constant velocity of movement of the fire front and the origin of time is chosen so that $t = 0$ corresponds to the initial position of the fire front, i.e., $X_f(0) = 0$.

Denote by $T(y)$ the time at which the fire construction group reaches the point $(f(y),y)$. Then

$$T(y) = \frac{1}{V_m(N/n)} \int_o^y \sqrt{1 + (f'(u))^2} \cdot du \tag{3}$$

and the time of control $T_c = T(L/n)$ is given by

$$T_c = \frac{1}{V_m(N/n)} \int_o^{L/n} \sqrt{1 + (f'(u))^2} \cdot du \quad .$$

Thus, the cost function (1) depends upon the firebreak path f(y) and the number of construction groups n as well as the total crew size N. Similarly, the last term of (1) depends upon n and an integral of f(y).

At any time t during the construction of the firebreak, the fire front cannot pass over the firebreak path f(y). Hence admissible firebreak paths f(y) are restricted to satisfy the constraints,

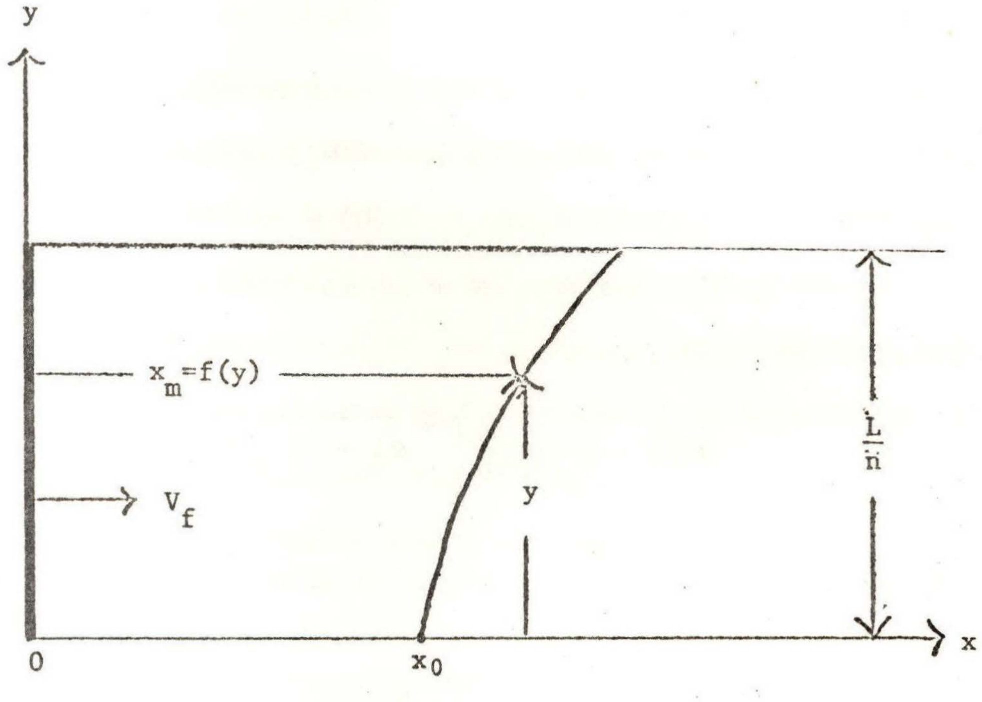


FIGURE 6. GEOMETRY OF PLANE WAVE FIREBREAK

$$(4)\quad f(y) \geq \frac{V_f}{V_m(N/n)} \int_0^y \sqrt{1 + (f'(u))^2} \cdot du \quad \text{for all } y \in [0, L/n] \quad .$$

In these variables the total cost function (1) becomes

$$K[f(y);\, n,\, N] = C_F + C_S \cdot N + \frac{C_m \cdot N}{V_m(N/n)} \int_0^{L/n} \sqrt{1 + (f'(y))^2} \cdot dy$$

(5)

$$+ C_B \cdot n \int_0^{L/n} f(y)dy$$

and extremal firebreaks are defined by those paths f(y) satisfying (4) which yield

$$(6)\qquad \underset{f(y)}{\text{Min}}\; K[f(y);\, N,\, n]$$

for given N, n.

5. The Optimization Procedure

For arbitrary positive crew size N and number of construction groups n, extremal firebreak paths are determined by the constrained variational problem (6), in which the endpoints are variable on the boundaries of the plane wave fire front. In this minimization, the extremal firebreak paths may either follow the fire front, be removed from the fire front, or be a mixture of these two possibilities.

Having determined the optimal firebreak path as a function of N and n, this function is substituted into (5) to obtain the total cost as a function of N and n, and the resulting cost function is then minimized with respect to the number of construction groups and total crew size.

Thus an optimal strategy is one which minimizes cost relative to crew size, number of groups, and firebreak path.

6. Optimal Firebreak for Given N and n

It is convenient to state certain general propositions concerning the extremal firebreak paths before undertaking the variational problem (6). They will be given without the proofs which are more or less self-evident.

Proposition 1: An extremal firebreak path may follow the fire front if and only if

$$\frac{V_f}{V_m(N/n)} < 1 \quad .$$

Proposition 2: If $\frac{V_f}{V_m(N/n)} < 1$ and the initial endpoint of the extremal firebreak path satisfies $f(0) = X_o = 0$, then the extremal path entirely follows the fire front. If $\frac{V_f}{V_m(N/n)} \geq 1$, there is no feasible control with this endpoint stipulation.

Corollary: An extremal firebreak path is a straight line follow-the-fire-front path starting from any value of y where fire front and firebreak meet on the extremal path.

These two propositions are required as preliminaries, because the standard variational procedures do not apply for a straight line follow-the-fire-front path which permits only one-sided variations.

With these preliminaries, there are evidently two parameter situations to be considered for extremal firebreak paths,

$$(a) \quad \rho = \frac{V_f}{V_m(N/n)} \geq 1$$

$$(b) \quad \rho = \frac{V_f}{V_m(N/n)} < 1 \quad .$$

For convenience let us refer to the problem of case (a) as Problem I and that of case (b) as Problem II.

$$\boxed{\text{Problem I: Case (a): } \rho \geq 1}$$

Here, following the fire front is not admissible, which suggests that the constraints (4) may be simplified to

$$(7) \qquad \int_0^{L/n} \left\{\rho \sqrt{1 + f'(y)^2} - f'(y) - \frac{f(0)}{L/n}\right\} dy = 0 \quad .$$

In fact, define

$$(8) \qquad G(y) = f(y) - \rho \int_0^y \sqrt{1 + f'(r)^2} dr$$

and

$$(9) \qquad G'(y) = f'(y) - \rho \sqrt{1 + f'(y)^2} \quad .$$

If $\rho \geq 1$, $G'(y) < 0$ for all $y \in [0, L/n]$ and the separation between men and fire front is a strictly decreasing function of y. Thus, condition (7) is a sufficient equality replacing (4), which is also obviously satisfied when f(y) is optimal.

Hence in this case we may formulate the variational problem as follows: Define

$$J(f(y)) = \int_{y_o}^{y_1} \left\{ L(y,\ f(y),\ f'(y)) - \frac{U_o f(y_o)}{y_1 - y_o} \right\} dy \tag{10}$$

where $y_o = 0$, $y_1 = L/n$, U_o is a multiplier, and

$$L(y,\ f(y),\ f'(y)) = \frac{C_m n}{V_m} \sqrt{1 + f'(y)^2} + C_B n f(y) + U_o[\rho \sqrt{1 + f'(y)^2} - f'(y)]\ . \tag{11}$$

Let $J(f(y))$ have an extremum for $f(y)$ and consider variations of $f(y)$ defined by $f(y) + h(y)$, where it is assumed that $f(y)$ and $h(y)$ are continuous and differentiable in $[0,\ L/n]$. As boundary conditions we apply $\delta X_1 = h(y_1)$, $\delta X_o = h(X_o)$, allowing the endpoints to vary on the lines $y = y_o$, $y = y_1$ as illustrated in Figure 7. For arbitrary $h(y)$ the increment in $J(f(y))$ is given by

$$\Delta J(f(y)) = \int_{y_o}^{y_1} \left\{ L(y,\ f(y) + h(y),\ f'(y) + h'(y)) - \frac{U_o}{(y_1 - y_o)}[f(y_o) + \delta X_o)] \right\} dy$$

$$- \int_{y_o}^{y_1} \left\{ L(y,\ f(y),\ f'(y)) - \frac{U_o}{(y_1 - y_o)} f(y_o) \right\} dy$$

and the corresponding variation δJ is

$$\delta J = \int_{y_o}^{y_1} \left\{ L_{f(y)} - \frac{d}{dy} L_{f'(y)} \right\} h(y)dy + L_{f'(y)}\Big|_{y=y_1} \cdot \delta X_1$$

$$- \left[L_{f'(y)}\Big|_{y=y_o} + U_o \right] \delta X_o$$

where $h(y)$, δX_1, and δX_o are arbitrary. Hence, as necessary conditions for $f(y)$ to minimize $J(f(y))$, we have

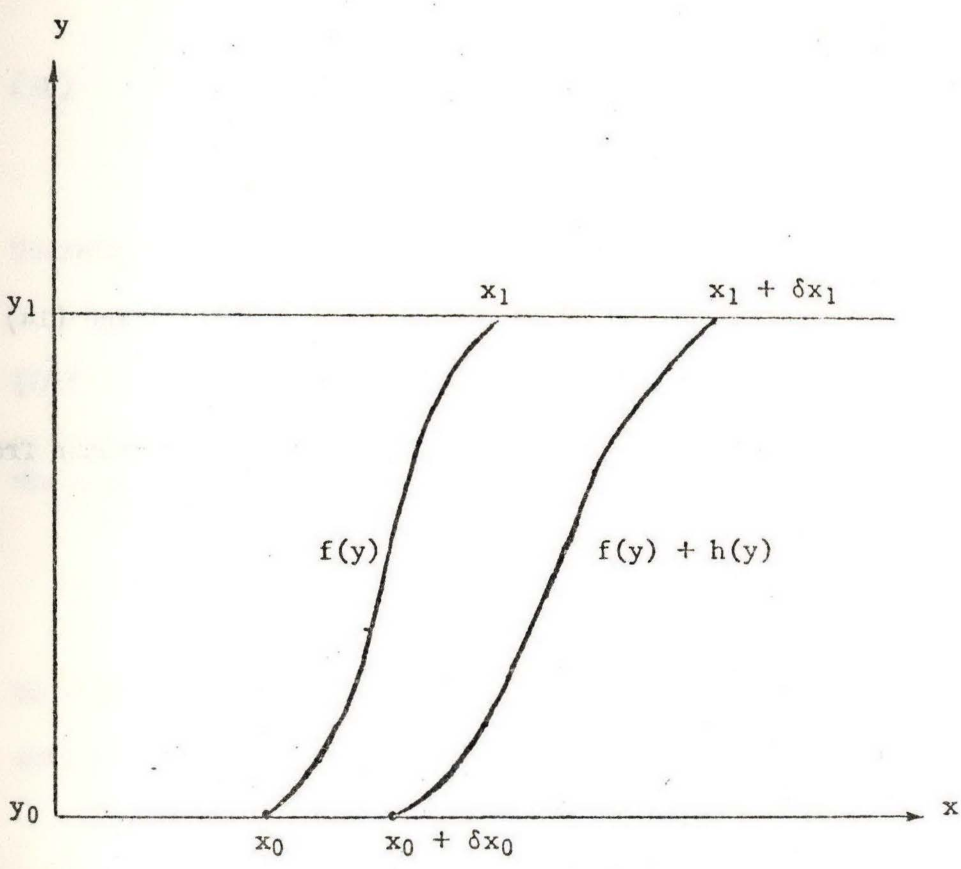


FIGURE 7. FIREBREAK VARIATIONS ($\rho \geq 1$)

(12) $$L_{f(y)} - \frac{d}{dy} L_{f'(y)} = 0$$

(13) $$L_{f'(y)}\Big|_{y=y_0} + U_0 = 0$$

(14) $$L_{f'(y)}\Big|_{y=y_1} = 0 \ .$$

Equation (12) is the standard Euler equation, while (13) and (14) are transversality conditions for the variable endpoints.

Now, using the definition of $L(y, f(y), f'(y))$, we obtain from (12)

(12.1) $$\frac{f'(y)}{\sqrt{1 + f'(y)^2}} = \alpha(U_o) \cdot y + \beta$$

where β is a constant of integration and

(15) $$\alpha(U_o) = \frac{C_B}{\frac{C_m}{V_m} + \frac{\rho}{n} U_o} \ .$$

Equation (13) becomes

(13.1) $$\frac{f'(0)}{\sqrt{1 + f'(0)^2}} = 0 \Rightarrow f'(0) = 0$$

and $\beta = 0$. Equation (14) states

(14.1) $$\left(\frac{C_m n}{V_m} + \rho U_o\right) \frac{f'(L/n)}{\sqrt{1 + f'(L/n)^2}} = U_o$$

and using (12.1) with $\beta = 0$ one obtains

$$(16) \qquad U_o = C_B \cdot L$$

and

$$(17) \qquad \alpha(U_o) = \alpha_o = \frac{C_B}{\frac{C_m}{V_m} + \rho C_B \frac{L}{n}}$$

Therefore the extremal firebreak path for case (a) is given by

$$(18) \qquad X_m = f(y) = X_o + \frac{1}{\alpha_o}\left\{1 - \sqrt{1 - (\alpha_o y)^2}\right\}$$

where $X_o = f(0)$. This last equation may be written

$$\left[X_m - \left(X_o + \frac{1}{\alpha_o}\right)\right]^2 + y^2 = \frac{1}{\alpha_o^2}$$

to show that the optimal firebreak path for given n, N is a circular arc with center $(X_o + 1/\alpha_o, 0)$ and radius $1/\alpha_o$, as illustrated in Figure 8.

The initial coordinate X_o is determined, from substitution of (18) into (7), to be

$$(19) \qquad X_o = \frac{\rho}{\alpha_o} \sin^{-1}\left(\frac{\alpha_o L}{n}\right) + \frac{1}{\alpha_o}\left[\sqrt{1 - \left(\frac{\alpha_o L}{n}\right)^2} - 1\right] .$$

The quantity $\alpha_o L/n$ is less than unity, since $\rho \geq 1$ for case (a) and $\alpha_o y < 1$ for all $y \in [0, L/n]$. Thus we have a complete solution for the extremal firebreak path.

One final remark: the cost function (5) is convex in f(y) and f'(y), and the necessary conditions (12), (13), (14) are sufficient for the determination of the optimal firebreak path for any given positive

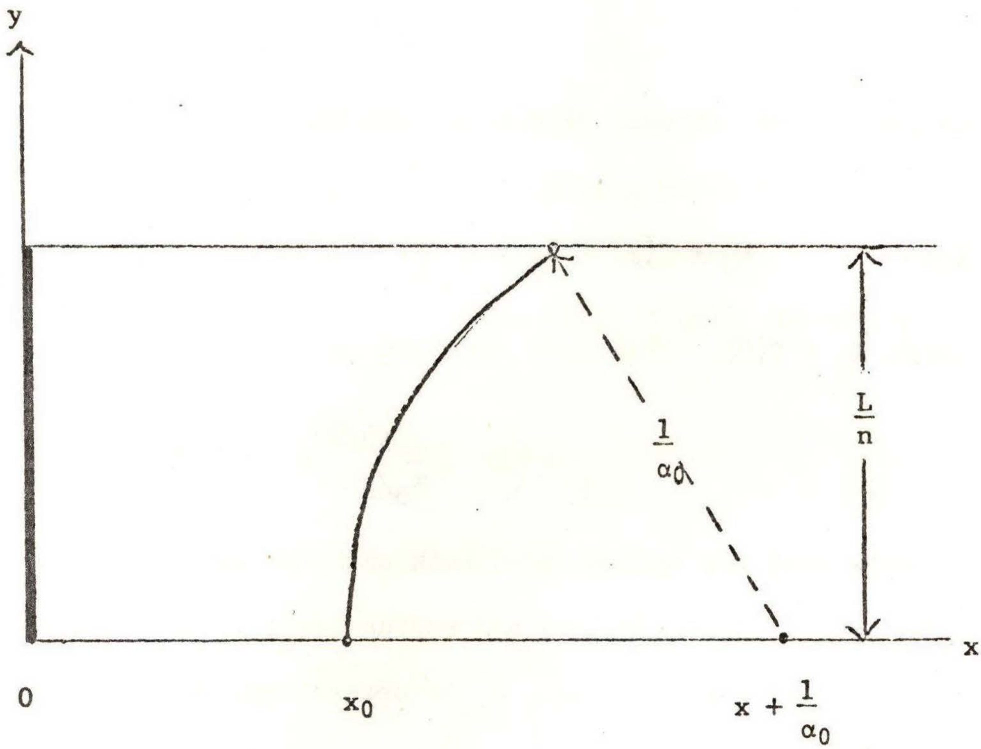


FIGURE 8. EXTREMAL FIREBREAK ($\rho \geqq 1$)

values of N and n [7].

$\boxed{\text{Problem II: Case (b): } \rho < 1}$

In this case it is possible for the firebreak to follow the fire front starting from any value of $y \in [0, L/n]$ (see Proposition 2 and the corollary following) and, in order to investigate the composite paths which may occur, let

$$(20) \qquad \tilde{y} = \text{Min } y \ni f(y) = \rho \int_0^y \sqrt{1 + f'(r)^2}\,dr \quad .$$

Then for $\tilde{y} \in [0, L/n]$ the variational problem for extremal firebreaks over the interval $[0, \tilde{y}]$ is one which minimizes (5) with (L/n) in the upper integral limits replaced by $\tilde{y}$. As in case (a) we shall replace the constraints (4) by

$$(7.1) \qquad \int_0^{\tilde{y}} \left\{ \rho \sqrt{1 + f'(y)^2} - f'(y) + \frac{f(0)}{\tilde{y}} \right\} = 0$$

and show that the solution satisfies (4) for all values of $y \in [0, \tilde{y}]$. The complete path of the optimal firebreak consists of this solution and a straight line with initial point $(f(\tilde{y}), \tilde{y})$ and slope $\rho/\sqrt{1 - \rho^2} = dx/dy$.

Hence for case (b) we use the definition (11) for $L(y, f(y), f'(y))$ and formulate the variational problem as follows: Define

$$J(f(y)) = \int_{y_o}^{\tilde{y}} \left\{ L(y, f(y), f'(y)) - \frac{U_o f(y_o)}{(\tilde{y} - y_o)} \right\} dy + \frac{C_m n}{V_m} \frac{(y_1 - \tilde{y})}{\sqrt{1 - \rho^2}}$$

$$+ C_B n \left\{ \tilde{X}(y_1 - \tilde{y}) + \frac{1}{2} (y_1 - \tilde{y})^2 \cdot \frac{\rho}{\sqrt{1 - \rho^2}} \right\}$$

(21)

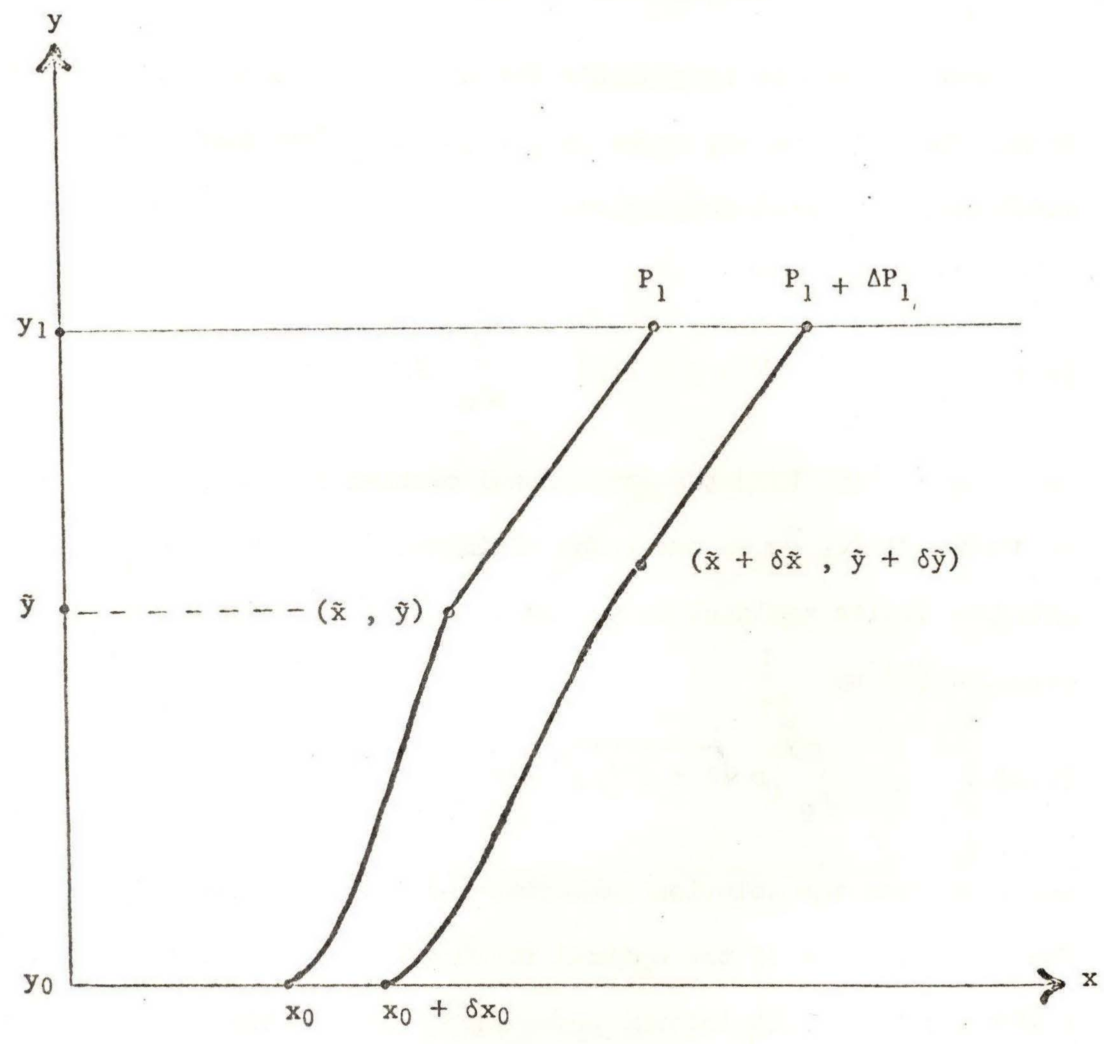


FIGURE 9. FIREBREAK VARIATIONS $(\rho < 1)$

taking a firebreak path $f(y)$ for $y \varepsilon [0, \tilde{y}]$ and a straight line follow-the-fire-front path for $y \varepsilon [\tilde{y}, y_1]$, where $y_o = 0$, $y_1 = L/n$, $\tilde{y} \varepsilon [0, L/n]$ and U_o is a multiplier. Let $J(f(y))$ have an extremum for $f(y)$ ($y \varepsilon [0, \tilde{y}]$) and consider variations of $f(y)$ defined by $f(y) + h(y)$, where it is assumed that $f(y)$ and $h(y)$ are continuous and differentiable in $[0, \tilde{y}]$. The boundary conditions are

$$\delta X_o = h(y_o)$$

$$\delta\tilde{X} - \frac{\rho}{\sqrt{1-\rho^2}} \cdot \delta\tilde{y} = h(\tilde{y}) \ ,$$

using a general variation $(\delta\tilde{x}, \delta\tilde{y})$ at the endpoint $(f(\tilde{x}), \tilde{y})$ as illustrated in Figure 9. Then for arbitrary $h(y)$ the increment in $J(f(y))$ is given by

$$\Delta J(f(y)) = \int_{y_o}^{\tilde{y}+\delta\tilde{y}} \left\{L(y,f(y)+h(y),f'(y)+h'(y)) - \frac{U_o}{(\tilde{y}+\delta\tilde{y}-y_o)} [f(y_o)+\delta X_o]\right\} dy$$

$$- \int_{y_o}^{\tilde{y}} \left\{L(y,f(y),f'(y)) - \frac{U_o}{(\tilde{y}-y_o)} f(y_o)\right\} dy + \frac{C_m n}{V_m} \frac{1}{\sqrt{1-\rho^2}} \left\{(y_1-\tilde{y}-\delta\tilde{y}) - (y_1-\tilde{y})\right\}$$

$$+ C_B n \left\{(\tilde{X}+\delta\tilde{X})(y_1-\tilde{y}-\delta\tilde{y}) + \tfrac{1}{2}(y_1-\tilde{y}-\delta\tilde{y})^2 \cdot \frac{\rho}{\sqrt{1-\rho^2}} - \tilde{X}(y_1-\tilde{y}) - \tfrac{1}{2}(y_1-\tilde{y})^2 \cdot \frac{\rho}{\sqrt{1-\rho^2}}\right\}$$

and the corresponding variation δJ is

$$\delta J = \int_{y_o}^{\tilde{y}} \left\{ L_{f(y)} - \frac{d}{dy} L_{f'(y)} \right\} h(y)dy$$

$$- \left\{ L_{f'(y)} \Big|_{y=y_o} + U_o \right\} \delta X_o + \left\{ C_B n(y_1 - \tilde{y}) + L_{f'(y)} \Big|_{y=\tilde{y}} \right\} \delta \tilde{X}$$

$$+ \left\{ L \Big|_{y=\tilde{y}} - \frac{C_m n}{V_m} \frac{1}{\sqrt{1-\rho^2}} - C_B n \left(\tilde{x} + (y_1 - \tilde{y}) \frac{\rho}{\sqrt{1-\rho^2}} \right) - L_{f'(y)} \Big|_{y=\tilde{y}} \frac{\rho}{\sqrt{1-\rho^2}} \right\} \delta \tilde{y}$$

where $h(y)$, δX_o, $\delta \tilde{X}$, and $\delta \tilde{y}$ are arbitrary. This problem is one of Bolza type [1]. Necessary conditions for $f(y)$ to minimize $J(f(y))$ are:

$$L_{f(y)} - \frac{d}{dy} L_{f'(y)} = 0 \tag{22}$$

$$L_{f'(y)} \Big|_{y=y_o} + U_o = 0 \tag{23}$$

$$L_{f'(y)} \Big|_{y=\tilde{y}} + C_B n(y_1 - \tilde{y}) = 0 \tag{24}$$

$$L \Big|_{y=\tilde{y}} - \frac{C_m n}{V_m} \frac{1}{\sqrt{1-\rho^2}} - C_B n\tilde{x} = 0 \quad . \tag{25}$$

Here, equations (23) and (24) are transversality conditions for the variable endpoints and (25) is a corner condition on the endpoint with general variation.

Now, using the definition of $L(y, f(y), f'(y))$ we obtain, as before in case (a),

$$\frac{f'(y)}{\sqrt{1 + f'(y)^2}} = \alpha(U_o)y + \beta . \tag{22.1}$$

Equation (23) has the same form as (13.1) and implies $\beta = 0$. Equation (24), although modified relative to (14), again implies $U_o = C_B L$ and $\alpha(U_o)$ is again defined by (17). Hence the extremal firebreak path for the interval $[0, \tilde{y}]$ is given by equation (18).

The corner condition (25) yields

$$\tilde{y} = \frac{\rho}{\alpha_o} \tag{26}$$

giving a determination of the endpoint $(f(\tilde{y}), \tilde{y})$. However, since $\rho < 1$, equation (26) does not surely determine a value for $\tilde{y}$ which is less than (L/n). But, using (18) for the extremal path in the cost function (5) with $\tilde{y}$ replacing (L/n), the resulting expression is convex in $\tilde{y}$ for any positive N and n. Hence the solution for $\tilde{y}$ may be written

$$\tilde{y} = \begin{cases} \dfrac{\rho}{\alpha_o} & \text{if} \quad \rho < \dfrac{\alpha_o L}{n} \quad \text{or} \quad n^2 < \left(\alpha_o L \dfrac{V_m}{V_f}\right) N \\[2ex] \dfrac{L}{n} & \text{if} \quad \rho \geq \dfrac{\alpha_o L}{n} \quad \text{or} \quad n^2 \geq \left(\alpha_o L \dfrac{V_m}{V_f}\right) N \end{cases} . \tag{27}$$

The initial coordinate X_o for case (b) is determined (by using (18), (27), and (7.1)) to be

$$X_o = \frac{\rho}{\alpha_o} \sin^{-1}(\rho) + \frac{1}{\alpha_o}\left[\sqrt{1 - \rho^2} - 1\right] \tag{28}$$

when $n^2 < \left(\alpha_o L \frac{V_m}{V_f}\right) N$; otherwise it is given by equation (19).

The question remains--does this solution satisfy the constraints (4)? Using (18) in (9), we obtain

$$G'(y) = \frac{\alpha_o(y - \frac{\rho}{\alpha_o})}{\sqrt{1 - (\alpha_o y)^2}}$$

and, since $y < \tilde{y} \leq \frac{\rho}{\alpha_o}$ for all $y \in [0, \tilde{y}]$, $G(y)$ is strictly decreasing in the interval $[0, \tilde{y}]$ to a zero value at $\tilde{y}$. Thus the constraints (4) are satisfied. Moreover $\alpha_o y \leq \rho < 1$ for all $y \in [0, \tilde{y}]$.

The general form of the optimal firebreak is illustrated in Figure 10, consisting of a circular arc connected to a straight line follow-the-fire-front piece with slope $\rho/\sqrt{1 - \rho^2}$, and at $y = \tilde{y} < L/n$ the slope $f'(\tilde{y})$ may be computed from (18) and (27) to have the same value. Hence the condition (25) is in effect a Weierstrass-Erdman corner condition [2].

Summary for Cases (a) and (b)

In Case (a), recall that $\rho \geq 1$ and $\alpha_o L/n < 1$. Hence it follows that

$$\rho \geq 1 \Rightarrow n \geq \frac{V_m N}{V_f}$$

$$\frac{\alpha_o L}{n} < 1 \Rightarrow n > \alpha_o L$$

and

$$n^2 \geq \alpha_o L \frac{V_m N}{V_f} .$$

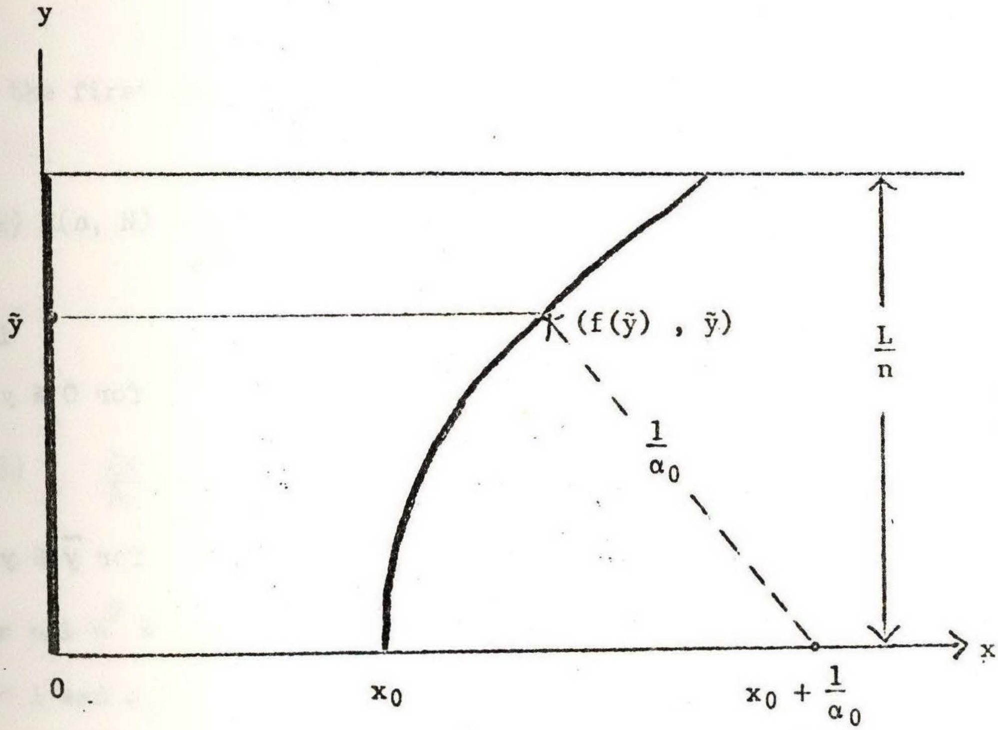


FIGURE 10. EXTREMAL FIREBREAK $(\rho < 1)$

ARBITRARY CORNER $\tilde{y}$

Thus, if we regard $\tilde{y}$ to be a variable for Case (a) as well as Case (b), we may state for both cases that

$$(29) \qquad \tilde{y} = \begin{cases} \dfrac{\rho}{\alpha_o} & \text{if } n^2 < \alpha_o L \dfrac{V_m N}{V_f} \\[2ex] \dfrac{L}{n} & \text{if } n^2 \geqq \alpha_o L \dfrac{V_m N}{V_f} \end{cases}$$

and the optimal firebreak path for given (N, n) is

$$(30) \quad f(y) = \begin{cases} X_o + \dfrac{1}{\alpha_o}\left\{1 - \sqrt{1 - (\alpha_o y)^2}\right\} & \text{for } 0 \leqq y \leqq \tilde{y} \\[2ex] X_o + \dfrac{1}{\alpha_o}\left\{1 - \sqrt{1 - \rho^2}\right\} + \dfrac{\rho}{\sqrt{1 - \rho^2}} \cdot y & \text{for } \tilde{y} \leqq y \leqq \dfrac{L}{n} \end{cases}$$

where $\rho = \dfrac{V_f}{V_m(N/n)}$ and

$$(31) \qquad X_o = \frac{\rho}{\alpha_o} \sin^{-1}(\alpha_o \tilde{y}) + \frac{1}{\alpha_o}\left\{\sqrt{1 - (\alpha_o \tilde{y})^2} - 1\right\}$$

$$\alpha_o = \frac{C_B}{\dfrac{C_m}{V_m} + \dfrac{C_B L V_f}{V_m N}}$$

7. <u>Optimal Number of Construction Groups and Optimal Crew Size</u>

For arbitrarily given crew size N, the total cost under optimal firebreak path depends upon the number of construction groups. Treating n continuously, we investigate the cost function K(n, N) and seek the

optimal number of construction groups n^* for arbitrary $N > 0$.

The function $K(n, N)$ has two forms depending upon whether

$$\text{Case (i)} \quad n^2 \geq \alpha_o L \left(\frac{V_m N}{V_f}\right)$$

$$\text{Case (ii)} \quad n^2 < \alpha_o L \left(\frac{V_m N}{V_f}\right) .$$

In the first case, it follows that,

$$K(n, N) = C_F + C_S \cdot N + \frac{C_B L}{2\alpha_o} \sqrt{1 - \left(\frac{\alpha_o L}{n}\right)^2} + \frac{C_B n}{2(\alpha_o)^2} \sin^{-1} \left(\frac{\alpha_o L}{n}\right) \tag{32}$$

and

$$\frac{\partial K}{\partial n} = \frac{C_B}{2(\alpha_o)^2} \left\{\sin^{-1} \left(\frac{\alpha_o L}{n}\right) - \left(\frac{\alpha_o L}{n}\right) \sqrt{1 - \left(\frac{\alpha_o L}{n}\right)^2} \right\} > 0 \tag{33}$$

for all $n^2 \geq \alpha_o L(V_m N/V_f)$, since this inequality implies $\alpha_o L/n < 1$ when $\rho < 1$ and $\alpha_o L/n < 1$ for $\rho \geq 1$ (see Case (a) above). Hence $K(n, N)$ is a monotone increasing function of n for all N in the range $n^2 \geq \alpha_o L(V_m N/V_f)$.

In the second case, it follows that

$$K(n, N) = C_F + C_S \cdot N + \left(\frac{C_m L}{V_m} + \frac{C_B L^2}{2n} \rho\right) \frac{1}{\sqrt{1 - \rho^2}} - \frac{C_B n}{2(\alpha_o)^2} \left[\frac{\rho}{\sqrt{1 - \rho^2}} - \sin^{-1} (\rho)\right] \tag{34}$$

and

$$\frac{\partial K}{\partial n} = \frac{C_B}{2(\alpha_o)^2} \sin^{-1} (\rho) - \frac{1}{2C_B} \left(\frac{C_m}{V_m}\right)^2 \frac{\rho}{(1 - \rho^2)^{3/2}} \tag{35}$$

$$(36) \qquad \frac{\partial^2 K}{\partial n^2} = \frac{\rho}{n\sqrt{1-\rho^2}} \left[\frac{C_B}{2(\alpha_o)^2} - \frac{1}{2C_B} \left(\frac{C_m}{V_m} \right)^2 \frac{(1 + 2\rho^2)}{(1-\rho^2)^2} \right]$$

for all $0 < n^2 < \alpha_o L \frac{V_m N}{V_f}$. Although (34) is not monotone, it can be shown that, since $\rho < 1$ in this case, $\frac{\partial^2 K}{\partial n^2} < 0$ whenever $\frac{\partial K}{\partial n} = 0$ and $\frac{\partial K}{\partial n}_{n=0+} = 0$. Thus over the range $\left(0 < n^2 < \alpha_o L \frac{V_m N}{V_f}\right)$ the function K(n, N) is quasi-concave. (See Appendix for a proof.) Further K(n, N) is continuous at $n^2 = \alpha_o L \frac{V_m N}{V_f}$ having a form as illustrated in Figure 11, and clearly the optimal number of construction groups (not restricted to integers) for given N is

$$(37) \qquad n^* = \sqrt{\alpha_o L \frac{V_m N}{V_f}} \ .$$

Then, except for integer disparity, it follows from (29) that the optimal firebreak path is entirely an arc of a circle.

Turning now to the optimization of crew size, the cost function K(N) resulting from (37) is

(38)
$$K(N) = C_F + C_S \cdot N + \frac{C_B L}{2\alpha_o} \sqrt{1 - \frac{\alpha_o L V_f}{V_m N}} \cdot \frac{C_B \sqrt{\alpha_o L \frac{V_m N}{V_f}}}{2(\alpha_o)^2} \sin^{-1} \left(\sqrt{\frac{\alpha_o L V_f}{V_m N}} \right)$$

where α_o depends upon N as given by (17). It is convenient to make a change of variable.

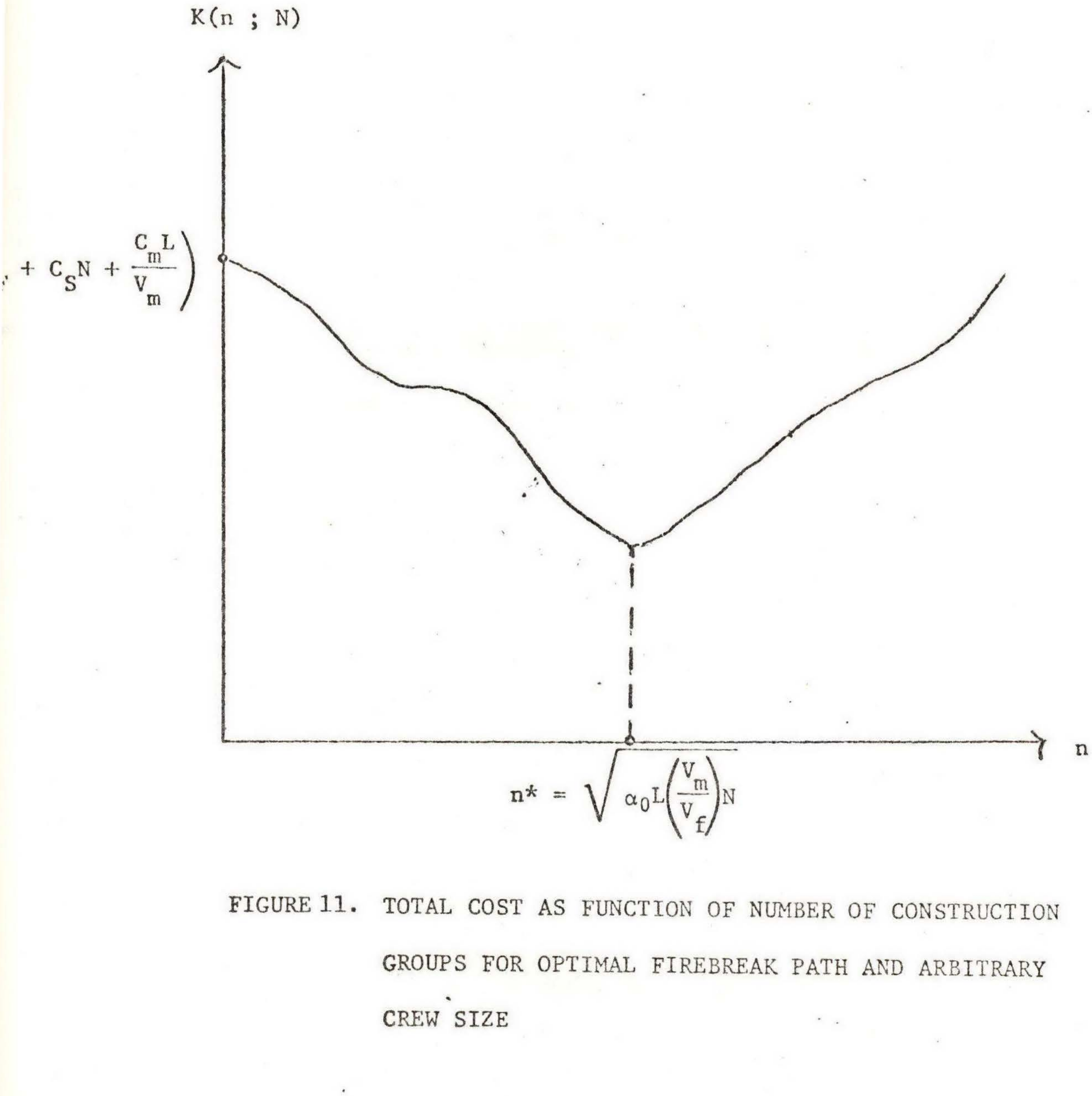


FIGURE 11. TOTAL COST AS FUNCTION OF NUMBER OF CONSTRUCTION GROUPS FOR OPTIMAL FIREBREAK PATH AND ARBITRARY CREW SIZE

$$\frac{1}{Q} = \frac{\alpha_o L V_f}{V_m N} = \frac{C_B L}{\frac{C_m}{V_f} N + C_B L} \quad . \tag{39}$$

Then the cost function K(N) is written

$$K(Q) = C_F + \frac{C_S}{C_m} V_f C_B L(Q - 1) + \frac{L}{2}\left(\frac{C_m}{V_m}\right)\left[\sqrt{\frac{Q}{Q - 1}} + \frac{Q^{3/2}}{(Q - 1)} \sin^{-1}\left(\frac{1}{\sqrt{Q}}\right)\right] \tag{40}$$

where Q is a monotone increasing function of N and $1 < Q < \infty$ as $0 < N < \infty$. Further

$$K'(Q) = \frac{C_S}{C_m} V_f C_B L - \frac{1}{4}\left(\frac{C_m}{V_m}\right)\left\{\frac{Q + 1}{\sqrt{Q}(Q - 1)^{3/2}} + \frac{\sqrt{Q}(3 - Q)}{(Q - 1)^2} \sin^{-1}\left(\frac{1}{\sqrt{Q}}\right)\right\} \quad . \tag{41}$$

Evidently K(Q) has a minimum for some finite value $Q^o > 1$, since $K''(Q)$ is positive over the range $1 < Q < \infty$ with $K'(Q) \to -\infty$ for $Q \to 1$ and $K'(Q) > 0$ for $Q \to \infty$.

Hence, treating N continuously in the range $0 < N < \infty$, the minimum of K(N) is obtained at a value

$$N^o = (Q^o - 1) C_B L \left(\frac{V_f}{C_m}\right) \tag{42}$$

and, if the maximum available number of men is $\overline{N}$ the optimal crew size is given by

$$N^* = \begin{cases} N^o & \text{if } N^o \leq \overline{N} \\ \overline{N} & \text{if } N^o > \overline{N} \end{cases} \tag{43}$$

With this optimal value of N, the optimal number of construction groups n^* is determined from (37) and in turn the optimal firebreak path is computed from equations (29), (30), and (31).

Note that this solution always yields a circular arc firebreak meeting the fire front during construction only at the endpoint $y = L/n$. But, if we restrict N and n to be integers such that (N/n) is an integer, the optimal firebreak may have a terminal straight line portion following the fire front.

In order to obtain the integral optimal values for crew size and number of construction groups, define

$$[a] = \text{largest integer equal to or less than } a$$

$$\langle a\rangle = \text{smallest integer equal to or greater than } a\ .$$

Then compare the following four costs:

$$\text{(i)}\quad K\left([n^*],\ [n^*]\cdot\left[\frac{N^*}{n^*}\right]\right)$$

$$\text{(ii)}\quad K\left([n^*],\ [n^*]\cdot\left\langle\frac{N^*}{n^*}\right\rangle\right)$$

$$\text{(iii)}\quad K\left(\langle n^*\rangle,\ \langle n^*\rangle\cdot\left[\frac{N^*}{n^*}\right]\right)$$

$$\text{(iv)}\quad K\left(\langle n^*\rangle,\ \langle n^*\rangle\cdot\left\langle\frac{N^*}{n^*}\right\rangle\right)$$

and select the corresponding integral optimal values for n and N.

Notice that the following subproblems which may be of interest in themselves have also been solved:

(1) Given N and n, finding an optimal firebreak path.

(2) Given N, finding the optimal number of construction groups and the optimal firebreak path for each.

(3) Given that $N \leqq \bar{N}$, finding the optimal values of N and n as well as the optimal path for the firebreaks.

It is interesting to observe that certain intuitive results for particular cases are obtained by substituting the relevant values of the parameters.

For example, when $C_S = 0$ (i.e., there are no transportation or "one-shot" logistic costs) and unlimited men are available, we obtain $N^* = \infty$, $n^* = \infty$, $X_o = 0$ and the optimal firebreak consists of a straight line which coincides with the initial position of the fire, recalling that we assumed the time required to get the fire fighters distributed along a working line was negligible. Moreover the related optimized total cost is finite, being

$$C_F + L \frac{C_m}{V_m}$$

obtained by computing $\lim_{Q \to \infty} K(Q)$ from (40).

Next consider $C_m = 0$, i.e., a situation where fire fighting is done by volunteers. Then, it follows from equations (33) and (35) that $n^* = 1$ for any N, and from (40) it follows that $N^* = 1$. Then from (29)

$$\tilde{y} = \begin{cases} \left(\frac{V_f}{V_m}\right)^2 & \text{if } V_f < V_m \\ L & \text{if } V_f \geqq V_m \end{cases}$$

and optimal firebreak paths of both types arise depending upon the relative size of V_f and V_m.

If $C_B = 0$, it is clear from the structure of the problem that $N^* = 0$ implying that it is optimal not to fight the fire.

CHAPTER III

AN ALTERNATIVE APPROACH USE OF CONTROL THEORY

1. Introduction

Problems I and II of Chapter II are evidently problems of optimal control theory. In Chapter II we chose to analyze these problems by the use of variational calculus and hence formulated the problems in a form convenient for the use of variational calculus. Here we shall recast the same problems in standard control theory notations and show how we can solve the same problems using the theory of optimal control. Finally we also compare the two approaches as far as this problem is concerned.

2. Formulation

The problem studied in Chapter II consisted of minimizing (5) subject to (4). Now if we define

(44) $$X_1(y) = f(y)$$

(45) $$X_2(y) = \int_o^y \sqrt{1 + [f'(y)]^2} \; dy$$

and

(46) $$\frac{dX_1(y)}{dy} = \dot{X}_1(y) = u(y) \quad ,$$

then minimizing (5) subject to (4) becomes equivalent to the minimizing (47) subject to (48)-(54).

(47) $$K = C_F + C_s \cdot N + \int_o^{L/n} \left[\left(\frac{C_m \frac{N}{}}{V_m \cdot \frac{N}{n}} \right) \sqrt{1 + [u(y)]^2} + C_B \cdot n \cdot X_1(y) \right] dy$$

$$\frac{dX_1(y)}{dy} = \dot{X}_1 = u(y) \tag{48}$$

$$\frac{dX_2(y)}{dy} = \dot{X}_2 = \sqrt{1 + [u(y)]^2} \tag{49}$$

$$\frac{V_f}{V_m(N/n)} X_2(y) - X_1(y) \leq 0 \tag{50}$$

$$- X_1(0) \leq 0 \tag{51}$$

$$- X_2(0) \leq 0 \tag{52}$$

$$- X_1(L/n) \leq 0 \tag{53}$$

$$- X_2(L/n) \leq 0 \tag{54}$$

Condition (50) is obtained by substituting (44), (45), and (46) in (4). Conditions (51)-(54) imply the nonnegativity of $f(y)$ and the arc length of the firebreak and are initial and end point conditions. Condition (49) follows directly from (44), (45), and (46).

3. Solution

The above problem, i.e., minimizing (47) subject to (48)-(54), is stated in the standard form used in control theory and one could apply Pontryagin's principle [6] to the above problem. But Pontryagin's principle is only a set of necessary conditions and does not assure a global optimum, which is what we seek.

But since the objective (47) is convex in $X_1(y)$, $X_2(y)$, and $u(y)$ and so are the constraints, we can apply Mangasarian's [3] conditions which subsume Pontryagin's conditions and are sufficient to assure a global optimal solution for our problem. We therefore quote Mangasarian's [3] conditions for convenience.

Consider the problem

$$\text{(55)} \qquad \text{Min } J = \int_{t_o}^{t_1} \phi(t, X(t), u(t))dt$$

Subject to

$$\text{(56)} \qquad \frac{dX}{dt} = \dot{X} = g(t, x(t), u(t))$$

$$\text{(57)} \qquad h(t, X(t), u(t)) \leqq 0$$

$$\text{(58)} \qquad p(X(t_o)) \leqq 0$$

$$\text{(59)} \qquad q(X(t_1)) \leqq 0$$

where X, u, g, h, p, q are vectors. Conditions (58) are initial point conditions and (59) are end point conditions.

<u>Theorem [3]</u>

Let $\phi(t, X, u)$ and each component of $g(t, X, u)$ and $h(t, X, u)$ be differentiable and convex in the variables (X, u) for $t \in [t_o, t_1]$ and let each component of $p(X(t_o))$ and $q(X(t_1))$ be differentiable and convex in $X(t_o)$ and $X(t_1)$ respectively. If there exist vectors $\bar{u}(t)$, $\bar{X}(t)$, $\bar{v}(t)$, $\bar{w}(t)$, $\bar{r}$, and $\bar{s}$ with $\bar{X}(t)$ and $\bar{v}(t)$ continuous and $\bar{w}(t)$ integrable and such that

$$(60) \quad \nabla_x \phi(t, \bar{X}, \bar{u}) + \nabla_x \bar{v}g(t, \bar{X}, \bar{u}) + \nabla_x \bar{w}h(t, \bar{X}, \bar{u}) + \dot{\bar{v}}(t) = 0$$

$$(61) \quad \nabla_u \phi(t, \bar{X}, \bar{u}) + \nabla_u \bar{v}g(t, \bar{X}, \bar{u}) + \nabla_u \bar{w}h(t, \bar{X}, \bar{u}) = 0$$

$$(62) \quad \nabla_{x(t_o)} \bar{r}p(\bar{X}(t_o)) + \bar{v}(t_o) = 0$$

$$(63) \quad \nabla_{x(t_1)} \bar{s}q(\bar{X}(t_1)) - \bar{v}(t_1) = 0$$

$$(64) \quad \bar{r} \geq 0$$

$$(65) \quad \bar{r}p(\bar{X}(t_o)) = 0$$

$$(66) \quad \bar{s} \geq 0$$

$$(67) \quad \bar{s}q(\bar{X}(t_1)) = 0$$

$$(68) \quad \bar{w}(t) \geq 0$$

$$(69) \quad \bar{w}(t)h(t, \bar{X}, \bar{u}) = 0$$

$$(70) \quad \bar{v}(t) \geq 0$$

then $\bar{u}(t)$, $\bar{X}(t)$ will minimize (55) subject to (56)-(59). Condition (70) need hold only for those components of $g(t, X, u)$ that are nonlinear in X or u or both.

<u>Remark</u>: Note that $\bar{v}(t)$, $\bar{w}(t)$, $\bar{r}$, $\bar{s}$ are multipliers for equations (56), (57), (58), and (59) respectively. Further conditions (60)-(70) are obtained by taking derivatives of the lagrangian with respect to variables and setting them equal to zero. Conditions (62), (63), (65), and (67) are transversality conditions; (69) is equivalent of Erdmann corner condition. Hence, on the whole, this approach does not seem to us too

much different from that used in Chapter II though such a distinction is made in the literature.

Using the above theorem to our problem, we get:

(60) $\Rightarrow$

$$\begin{bmatrix} C_B \cdot n \\ 0 \end{bmatrix} + \begin{bmatrix} -\bar{w}(y) \\ \frac{V_f}{V_m(N/n)}\bar{w}(y) \end{bmatrix} + \dot{\bar{v}}(y) = 0 \tag{71}$$

(61) $\Rightarrow$

$$\frac{\bar{u}(y)}{\sqrt{1+[\bar{u}(y)]^2}} = \frac{-\bar{v}_1(y)}{\frac{C_m \cdot n}{V_m} + \bar{v}_2(y)} \tag{72}$$

(62) $\Rightarrow$

$$\begin{bmatrix} -\bar{r}_1 \\ \bar{r}_3 - \bar{r}_2 \end{bmatrix} + \bar{v}(0) = 0 \tag{73}$$

(63) $\Rightarrow$

$$-\bar{s}_1 + \bar{v}_1(L/n) = 0 \tag{74}$$

(65) $\Rightarrow$

$$-\bar{r}_1\bar{X}_1(0) + (\bar{r}_3 - \bar{r}_2)\bar{X}_2(0) = 0 \tag{75}$$

(67) $\Rightarrow$

$$\bar{s}_1\bar{X}_1(L/n) = 0 \Rightarrow \bar{s}_1 = 0 \quad \text{since } \bar{X}_1(L/n) > 0 \tag{76}$$

(69) $\Rightarrow$

$$\bar{w}(y)\left[\frac{V_f}{V_m(N/n)}\bar{X}_2(y) - \bar{X}_1(y)\right] = 0 \tag{77}$$

Combining equations (63) and (76), we get

$$\bar{v}_1(L/n) = 0 \quad . \tag{78}$$

For Problem I, $\rho \leqq 1$ and hence men and fire meet only at the end at $y = L/n$. Hence (77) implies

$$\bar{w}(y) = 0 \qquad \text{for all} \quad y < L/n \tag{79}$$

Let us put $\bar{w}(y) = 0$ for all y. Then using (71), (73), and (75) we get

$$\dot{\bar{v}}_2(y) = 0 \tag{80}$$

$$\dot{\bar{v}}_1(y) = - C_B \cdot n \tag{81}$$

Combining (80) and (73)

$$v_2(y) = \text{constant} = \bar{r}_2 - \bar{r}_3 \tag{82}$$

and using (81) and (73) we get

$$v_1(y) = - C_B \cdot n \cdot y \tag{83}$$

Now substituting (82) and (83) in (72) we get

$$\frac{\bar{u}(y)}{\sqrt{1 + [\bar{u}(y)]^2}} = \frac{C_B \cdot n}{\frac{C_m \cdot n}{V_m} + (\bar{r}_2 - \bar{r}_3)} \cdot y \quad . \tag{84}$$

At this stage we are left with an unknown constant $(\bar{r}_2 - \bar{r}_3)$ which we have to determine. In order to determine this we have to substitute the results of equation (84) in the cost function (47) and then optimize over this constant, which is a tedious process. Hence, as compared to the previous analysis in Chapter II, which gave us the closed form solution directly, this seems to involve a further optimization problem. But

since we know the solution of the problem, instead of doing the optimization, we check whether our solution of Chapter II satisfies all conditions here.

Thus we let

$$\bar{r}_2 - \bar{r}_3 = C_B \cdot L \cdot \rho / n$$

and it can be readily seen that this satisfies all conditions of the problem and hence is the optimal solution.

4. Comparison of the Two Methods

It seems that wherever we can use one of the approaches, the other can also be used. In both methods we are setting derivatives of the lagrangian with respect to variables equal to zero and solving the resulting system of equations. The convexity of the objective and the constraints assume the global minimum in both methods and hence it seems that there is not an appreciable difference in the methods except for notations used. Further, the approach of optimal control theory leaves us with an unknown constant which is to be determined by a process of optimization of the cost function with respect to this variable. This involves additional work which is bypassed in the variational approach. Thus the direct variational approach seems to be more efficient for this problem.

CHAPTER IV

A STOCHASTIC MODEL

1. Introduction

So far we have assumed that the velocity of the fire is a known quantity. For fires of considerable magnitude, since time to control is quite large, there may be fluctuations in the velocity of fire due to changes in weather conditions. These changes would be unknown, thus leading us to consider the velocity of fire to be a random variable. In any case, such a random variable would only take on nonnegative values, and we can further assume that there is an upper bound on the realized values of this random variable. Such an assumption could easily be justified on practical grounds. We then minimize the total expected cost to determine the optimal strategy.

2. Assumptions to Reduce to a Succession of Deterministic One-Period Models

We shall be using a multistage dynamic programming decision model in which the dispatching of firebreak construction crew is done periodically based on the observed values of the velocity of fire at predetermined points of time. As in the deterministic case, the total crew will be made up of n groups with the further assumption that each one nth of the fire-front has some realized value of the fire velocity V_f at the beginning of any period. Such an assumption seems reasonable if the randomness in fire velocity is mainly due to changes in weather conditions since we ignore topographical peculiarities. The value of n is taken fixed for all periods, in order to determine the optimal policy for any number of construction groups and the resulting suboptimized cost function K(n) may

be minimized to determine the optimal number of construction groups n^*. Since in practice the values of n are small, computation of values of $K(n)$ for n up to 2 or 3 would suffice. An analysis of determining the optimal value of n seems very tedious.

For arbitrary n, the optimal policy is a feedback control rule which minimizes the total expected cost for an unknown number of periods, in which an equal number of men are dispatched to each construction group at the beginning of each review period, based upon the realized values of fire velocity in all previous periods. At this point we would like to state that no assumption is made about the nature of the fire velocity random variables of the different periods such as independence, or having stationary distributions, etc. The only assumption is that there is an upper bound for the realized values of all these random variables which is denoted by $\hat{V}_f$. But we assume that the velocity of fire is constant within a period and is the realized value of the random variable at the beginning of that period. Such an assumption is justified if we assume the periodic review time T is small and the whole model is not a good one for the case when the review time is large. In the case when T is large, the very idea of having periodic review, i.e., to gain some more information regarding the nature of the fire, seems to be lost. Hence, we can justify our assumption regarding small values of T.

Since it takes some time to send men to the actual spot where the fire is, it is reasonable to assume that a decision to send men in any period is taken before observing the fire velocity in that period and hence the optimal crew size is determined by the realized values of fire velocity in the previous periods only.

In practice, the optimal firebreak path during any period is determined in accordance with the realized value of the fire velocity at the beginning of the period and with the terminal point of the previous period as the initial point for the firebreak in the current period. The policy structure is open end, i.e., no fixed number of review periods are set in advance, and the terminating stage is reached when the men enclose the

fire front. For the extremal problem the main difference relative to the deterministic case is that the variational problem has a fixed initial point for all periods except the first. The initial point of the firebreak in the first period is a free end point of the related variational problem.

Since the initial points for all periods except the first are fixed, we impose the condition that crew size N for all periods except the first is large enough to follow the fire, i.e., $N_i \geq \hat{N}$ for $i \geq 2$ where $\hat{N}$ is defined by $\hat{N} = \hat{V}_f \cdot n/V_m$. Such an assumption is found not to be too restrictive because the practical values of $\hat{N}$ are small and the mathematical analysis simplifies a great deal. Such a restriction is not necessary for the first period since the initial point is free and we can always shift the whole firebreak curve to avoid following the fire.

Since the transportation cost C_s is incurred the instant a man is sent and the total cost is otherwise a nonincreasing function of the crew size, it is never optimal to call men back until the completion of the firebreak. Hence the crew size is nondecreasing over successive reviews.

First we shall consider a one-period model and then extend it to a multiperiod analysis and finally give a stopping rule, for computational purposes to determine the optimal number of periods.

3. One-Period Stochastic Model

Since the optimal firebreak path is determined on the basis of the realized value of the fire velocity, there is no change between this model and the deterministic model as far as the variational problem is concerned. Hence the total cost $K(n; N|V_f)$ for arbitrary n, N and a

realized value V_f of fire velocity is given by

$$K(n;\ N|V_f) = \begin{cases} C_F + C_S \cdot N + \dfrac{C_B \cdot L}{2\alpha_o}\sqrt{1 - \left(\dfrac{\alpha_o L}{n}\right)^2} + \dfrac{C_B \cdot L}{2\alpha_o^2} \sin^{-1}\left(\dfrac{\alpha_o L}{n}\right) & \text{for } \alpha_o L/n \geqq \rho \\[2ex] C_F + C_S \cdot N + \left(\dfrac{C_m L}{V_m} + \dfrac{C_B L^2 \rho}{2n}\right)\dfrac{1}{\sqrt{1 - \rho^2}} - \dfrac{C_B \cdot n}{2\alpha_o^2}\left\{\dfrac{\rho}{\sqrt{1 - \rho^2}} - \sin^{-1}\right. & \text{for } \alpha_o L/n < \rho \end{cases}$$

$$(85) \qquad \triangleq C_S \cdot N + G(n;\ N|V_f)$$

<u>Proposition 1</u>.

$G(n;\ N|V_f)$ is a nonincreasing function of N.

<u>Proof</u>. Let $N_1 > N_2$ and let $X = f_2^*(y)$ be the optimal firebreak path for N_2. Then $f_2^*(y)$ is a feasible path for N_1 since it satisfies (4) with $N = N_1$. Therefore, for $N = N_1$, there exists a feasible path with same cost as for $N = N_2$. Hence the proposition follows.

We shall from now on impose the condition that $N \geqq 1$, which is really no restriction.

<u>Definition</u>:

A function $G(N)$ is δ-convex iff

$$G(N + a) - G(N) - a\left[\frac{G(N) - G(N - b)}{b}\right] + \delta \geqq 0 \quad \text{for all } a \geqq 0,\ b > 0$$

<u>Proposition 2</u>.

Let $G(n;\ N_o|V_f) = \delta(n,\ N_o|V_f)$. Then $G(n,\ N|V_f)$ and hence $K(n;\ N|V_f)$ is $\delta(n;\ N_o|V_f)$-convex in N for all $N \geqq N_o$.

Proof:

$$\left\{G(n;N + a)|V_f) - G(n;N|V_f) - a\left[\frac{G(n;N|V_f) - G(n;N-b|V_f)}{b}\right] + \delta(n;N_o|V_f)\right\} \geq 0$$

for all $a \geq 0$, $b > 0$, $N \geq N_o$

because

$$G(n;\ N|V_f) - G(n;\ N - b|V_f) \leq 0$$

and

$$G(n;\ N|V_f) \leq \delta(n;\ N_o|V_f) = G(n;\ N_o|V_f) \quad \text{for all } N \geq N_o$$

Let

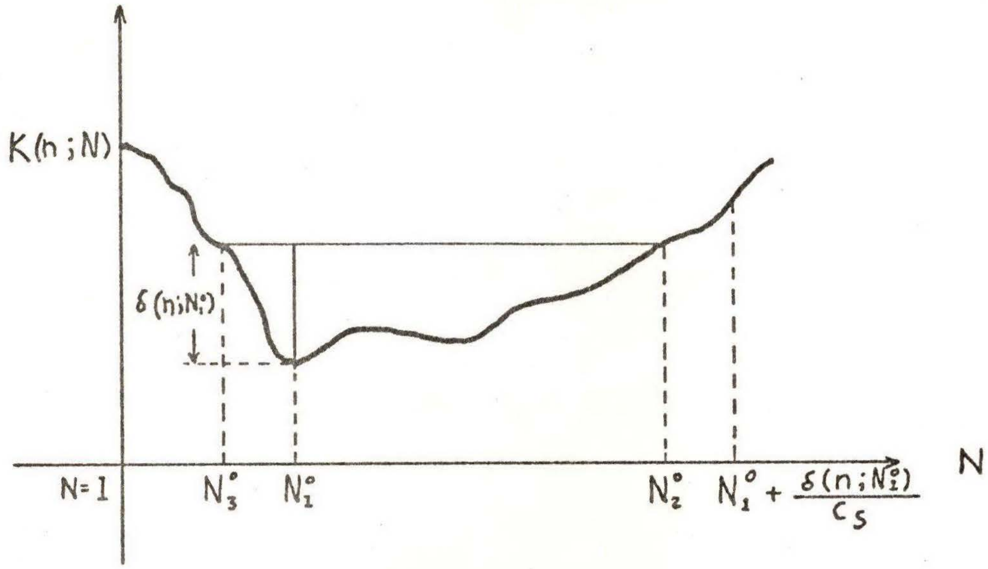
$$K(n;\ N) = E_{V_f}\left[K(n;\ N|V_f)\right] = \int_o^{\hat{V}_f} K(n;\ N|X)d\Phi_1(X)$$

$$\delta(n;\ N_o) = E_{V_f}\left[\delta(n;\ N_o|V_f)\right] = \int_o^{\hat{V}_f} \delta(n;\ N_o|X)d\Phi_1(X)$$

$$G(n;\ N) = E_{V_f}\left[G(n;\ N|V_f)\right] = \int_o^{\hat{V}_f} G(n;\ N|X)d\Phi_1(X)$$

where $\Phi_1(X)$ is the probability distribution function of the fire velocity random variable. Then from proposition 2, it follows that $K(n;\ N)$ is $\delta(n;\ N_o)$-convex in N for $N \geq N_o$. Now we can use these results to determine N^*, the optimum value of N.

Procedure for determining N*:



Let N_1° be the largest value of N such that K(n; N) is nonincreasing for $1 \leqq N \leqq N_1^{\circ}$. Further, let N_2° be the smallest value of $N \geqq N_1^{\circ}$ such that

$$K(n;\ N_2^{\circ}) = K(n;\ N_1^{\circ}) + \delta(n;\ N_1^{\circ}) \quad .$$

Then, by the property of δ-convexity, N_1^* which yields the global minimum of K(n; N) must lie in $[N_1^{\circ}, N_2^{\circ}]$. For, otherwise, the secant line through $(N_3^{\circ}, N_2^{\circ})$ would be more than δ above the K(n; N) curve at the point which yields the global minimum. Further, we shall now show that N_1^* lies within some interval which may be larger than $(N_1^{\circ}, N_2^{\circ})$ but is easier to compute.

<u>Proposition 3</u>

$$N_1^* \varepsilon \left[N_1^{\circ}, N_1^{\circ} + \frac{\delta(n; N_1^{\circ})}{C_s}\right]$$

<u>Proof</u>: Clearly $N_1^* \geq N_1^{\circ}$ by the definition of N_1°. We shall now show $N_1^* \leq N_1^{\circ} + \delta(n; N_1^{\circ})/C_s$

$$K(n; N) = C_s \cdot N + G(n; N)$$

$$K(n; N) - K(n; N_1^{\circ}) = C_s(N - N_1^{\circ}) + G(n; N) - G(n; N_1^{\circ})$$

$$= G(n; N) + C_s\left[(N - N_1^{\circ}) - \frac{G(n; N_1^{\circ})}{C_s}\right]$$

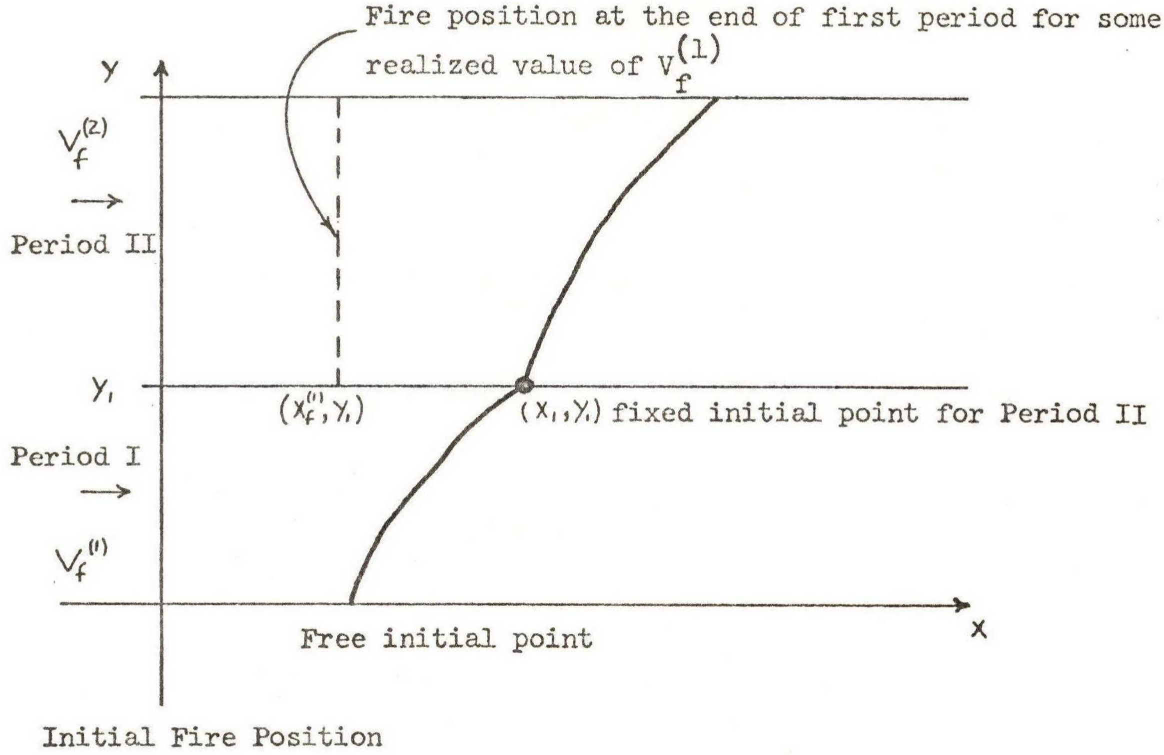
$$= G(n; N) + C_s\left[N - \left(N_1^{\circ} + \frac{\delta(n; N_1^{\circ})}{C_s}\right)\right]$$

$$\geq 0 \quad \text{for all } N \geq N_1^{\circ} + \frac{\delta(n; N_1^{\circ})}{C_s}$$

Hence the proposition is proved

Thus we enumerate the values of $K(n; N)$ relative to N for the interval $[N_1^{\circ}, N_1^{\circ} + \delta(n; N_1^{\circ})/C_s]$ and this involves very little additional work since $\delta(n; N_1^{\circ})$ would already have been calculated while enumerating the values of $K(n; N)$ from $N = 1$ up to $N = N_1^{\circ}$. The value of $\delta(n; N_1^{\circ})$ will be small and hence the above interval is also quite small. Thus we do not have to enumerate the values of $K(\cdot)$ with respect to N_1 much more than that required for determining N_1°.

4. Two-Period Problem



We shall now consider a two-period problem. Here, the initial point of the first period is free but the period is of a fixed duration of T units of time, whereas in the second period the initial point is fixed but we suppose that the firebreak goes up to $y = L/n$ during the next interval T. Further, at the start of the second period we are given the number of construction groups n and the crew size N_1 used to fight the fire in the first period. For arbitrary values of n, N_1 and the second period initial point (X_1, y_1) satisfying $X_1 \geq X_f^{(1)}$, we shall first determine the optimal firebreak path and optimal crew size for the second period. Then, using this optimized cost of the second period, determine the optimal policy for the first period.

Thus we shall first consider the optimization for Period II.

(a) Variational Problem for Period II

We can distinguish two cases: (i) $X_f^{(1)} = X_1$, i.e., the men and the fire meet sometime during the first period, and (ii) $X_f^{(1)} < X_1$, i.e., the men and the fire have not met till the beginning of the second period. We shall treat these two cases separately.

Case (i) $X_f^{(1)} = X_1$

In this case, by proposition 2 of Chapter II, the extremal path is one given by following the fire. Hence the total cost of period II, for any realized value of fire velocity is given by

$$K_2[n; N_1, N_2, X_1, y_1, X_f^{(1)} | V_f^{(1)}, V_f^{(2)}]$$

(86)
$$= C_F + C_S(N_2 - N_1) + C_B \cdot n\left(\frac{L}{n} - y_1\right)X_1 + \frac{C_m \cdot n}{V_m}\frac{\left(\frac{L}{n} - y_1\right)}{\sqrt{1 - \rho_2^{\,2}}} + \frac{C_B \cdot n \cdot \rho_2}{2}\frac{\left(\frac{L}{n} - y_1\right)^2}{\sqrt{1 - \rho_2^{\,2}}}$$

where

(87)
$$\rho_2 = \frac{V_f^{(2)} \cdot n}{V_m \cdot N_2} < 1 \quad \text{since } N_2 \geq \hat{N}$$

and $V_f^{(i)}$ = realized value of fire velocity for the ith period, i = 1, 2.

Remark:

It can be easily shown that $\frac{\partial^2 K_2}{\partial N_2^{\,2}}[n; N_1, N_2 \cdots | V_f^{(1)}, V_f^{(2)}] \geq 0$

and hence $K_2[n; N_1, N_2, X_1, y_1, X_f^{(1)} | V_f^{(1)}, V_f^{(2)}]$ is convex in N_2 for any realized value of $V_f^{(2)} \geq 0$.

Definition:

$$K_2[n;\, N_1,\, N_2,\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)}]$$

$$\triangleq E_{V_f^{(2)}}[K_2(n;\, N_1,\, N_2,\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)},\, V_f^{(2)})]$$

$$= \int_0^{\hat{V}_f} K_2[n;\, N_1,\, N_2,\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)},\, X]d\Phi_2(X)$$

where

$$\Phi_2(X) = \Pr\left\{V_f^{(2)} \leq X|V_f^{(1)}\right\}$$

By using the above remark and definition, we can show that $K_2[n;\, N_1,\, N_2,\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)}]$ is convex in N_2. Hence N_2^*, the optimal crew size for period II, is given by

$$N_2^* = \mathrm{Max}[N_1,\, N_2^{\,o},\, \hat{N}] \tag{88}$$

where

$N_2^{\,o}$ yields the global minimum of $K_2[n;\, N_1,\, N_2,\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)}]$

and

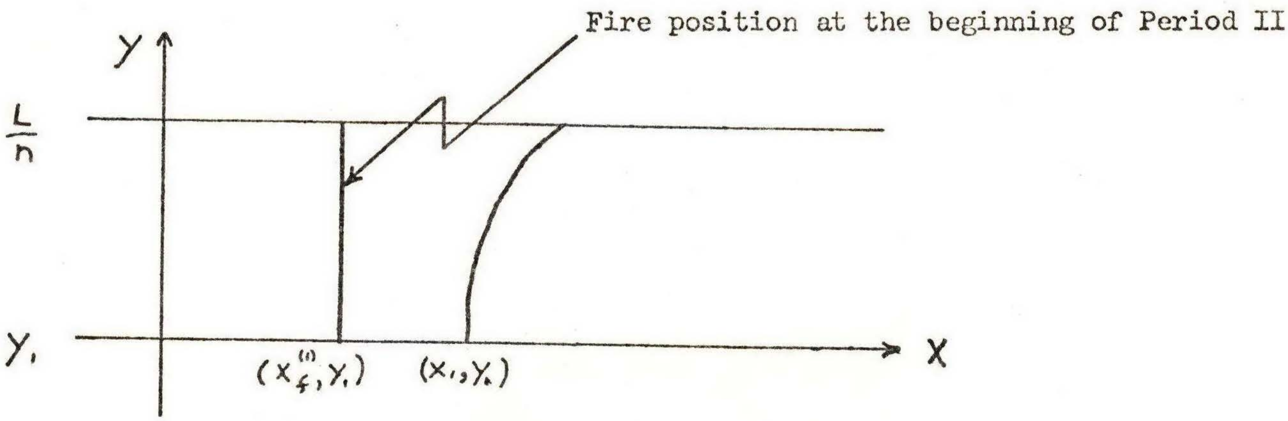
$$\hat{N} = \hat{V}_f \cdot n/V_m$$

$\hat{V}_f$ being the upper bound on the realized values of the fire velocity random variables.

Remark: We have already stated that it is never optimal to recall men, and hence $N_2^* \geq N_1$. Further, we also have stated that we will impose the condition that the crew size for all periods except the first be large

enough to follow the fire. $\hat{N}$ is the smallest number of men required to follow the fire and thus $N_2^* \geq \hat{N}$.

Case (ii) $X_f^{(1)} < X_1$



Proceeding on lines similar to those in Chapter II, the variational problem for this case can be stated as follows:

$$\underset{f_2(y)}{\text{Min}} \left\{ K_2[n; N_1, N_2, X_1, y_1, X_f^{(1)}, f_2(y) | V_f^{(1)}, V_f^{(2)}] \right\} \text{ subject to (90) and (91)}$$

where

$$K_2[n; N_1, N_2, X_1, y_1, X_f^{(1)}, f_2(y) | V_f^{(1)}, V_f^{(2)}]$$

$$= C_F + C_S(N_2 - N_1) + \frac{C_m \cdot n}{V_m} \int_{y_1}^{L/n} \{1 + (f_2'(y))^2\}^{1/2} dy + C_B \cdot n \int_{y_1}^{L/n} f_2(y) dy$$

(89)

$$f_2(y) \geq \left\{\rho_2 \int_{y_1}^{y} \left\{1 + (f_2'(y))^2\right\}^{1/2} dy\right\} + X_f^{(1)} \quad \text{for all } y \in [y_1, L/n]$$

(90)

and

(91) $$f_2(y_1) = X_1$$

We note that the condition (91) makes this problem differ from those considered in Chapter II. Also, since $N_2 \geq \hat{N}$, $\rho_2 < 1$ and we can follow the fire.

Let us define $\tilde{y}$ as follows:

$$\tilde{y} = \text{Min } y \geq y_1 \quad \text{such that}$$

(92) $$f_2(\tilde{y}) = X_f^{(1)} + \rho_2 \int_{y_1}^{\tilde{y}} \left\{1 + (f_2'(y))^2\right\}^{1/2} dy$$

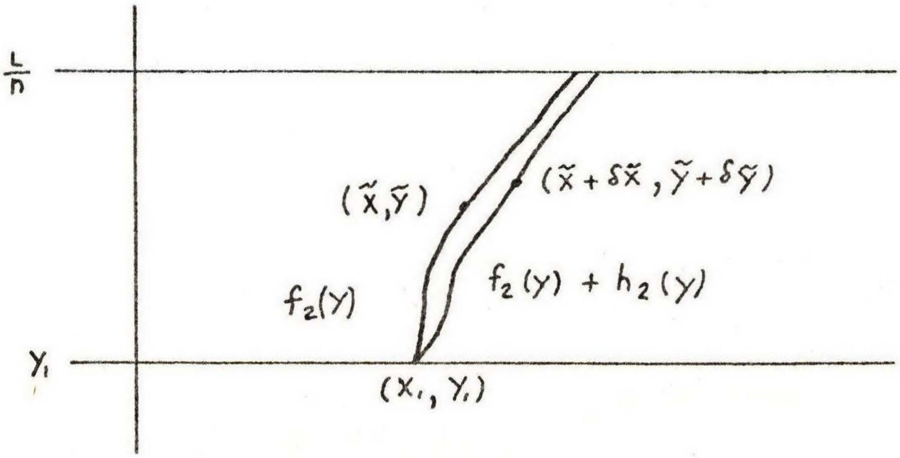
Then for $y \in [y_1, \tilde{y}]$ the variational problem is the same as minimizing (89) subject to (90) and (91) with L/n replaced by $\tilde{y}$ and the rest of the extremal path is a straight line with initial point $(f(\tilde{y}), \tilde{y})$ and slope $\frac{\sqrt{1 - \rho_2^2}}{\rho_2}$. Hence (90) can be replaced by

(93) $$\int_{y_1}^{\tilde{y}} \left\{\rho_2 \sqrt{1 + (f_2'(y))^2} - f'(y) - \frac{f(y_1) - X_f^{(1)}}{\tilde{y} - y_1}\right\} dy = 0$$

Define

$$J(f_2(y)) = \int_{y_1}^{\tilde{y}} \left\{ L(y, f_2(y), f_2'(y)) - U_o \frac{(f_2(y_1) - X_f^{(1)})}{\tilde{y} - y_1} \right\} dy$$

$$+ \frac{C_m \cdot n}{V_m} \frac{\frac{L}{n} - \tilde{y}}{\sqrt{1 - \rho_2^2}} + C_B \cdot n \cdot \left\{ \tilde{X}\left(\frac{L}{n} - \tilde{y}\right) + \frac{1}{2}\left(\frac{L}{n} - \tilde{y}\right)^2 \frac{\rho_2}{\sqrt{1 - \rho_2^2}} \right\}$$



Firebreak variations for $\rho_2 < 1$ and fixed initial point

Now we shall consider variations of the type $f_2(y) + h_2(y)$ where

$$\delta(X_1) = h_2(y_1) = 0$$

and $d\tilde{X}$ and $\delta\tilde{y}$ satisfy

$$\delta\tilde{X} - \frac{\rho_2}{\sqrt{1 - \rho_2^2}} \cdot \delta\tilde{y} = h_2(\tilde{y})$$

and $h_2(y)$ is continuous and differentiable.

$$\Delta J(f_2(y)) = \int_{y_1}^{\tilde{y}+\delta\tilde{y}} \left\{ L(y, f_2(y) + h_2(y), f_2'(y) + h_2'(y)) - U_o \frac{(f_2(y_o) - X_f^{(1)})}{(\tilde{y} + \delta\tilde{y} - y_1)} \right\} dy$$

$$- \int_{y_1}^{\tilde{y}} \left\{ L(y, f_2(y), f_2'(y)) - U_o \frac{(f_2(y_o) - X_f^{(1)})}{\tilde{y} - y_1} \right\} dy$$

$$+ \frac{C_m \cdot n}{V_m} \frac{1}{\sqrt{1 - \rho_2^2}} \left\{ \left(\frac{L}{n} - \tilde{y} - \delta\tilde{y}\right) - \left(\frac{L}{n} - \tilde{y}\right) \right\}$$

$$+ C_B \cdot n \left[(\tilde{X} + \delta\tilde{X})\left(\frac{L}{n} - \tilde{y} - \delta\tilde{y}\right) + \frac{1}{2}\left(\frac{L}{n} - \tilde{y} - \delta\tilde{y}\right)^2 \frac{\rho_2}{\sqrt{1 - \rho_2^2}} \right.$$

$$\left. - \tilde{X}\left(\frac{L}{n} - \tilde{y}\right) - \frac{1}{2}\left(\frac{L}{n} - \tilde{y}\right)^2 \frac{\rho_2}{\sqrt{1 - \rho_2^2}} \right]$$

and

$$\delta J = \int_{y_1}^{L/n} \left\{ L_{f_2(y)} - \frac{d}{dy}\left(L_{f_2'(y)}\right) \right\} h_2(y)dy + \left[L_{f_2'(y)}\Big|_{y=\tilde{y}} + C_B \cdot n \left(\frac{L}{n} - \tilde{y}\right) \right] \delta\tilde{X}$$

$$+ \left\{ L\Big|_{y=\tilde{y}} - \frac{C_m \cdot n}{V_m} \frac{1}{\sqrt{1 - \rho_2^2}} - C_B \cdot n \cdot \left(\tilde{X} + \left(\frac{L}{n} - \tilde{y}\right) \frac{\rho_2}{\sqrt{1 - \rho_2^2}} \right. \right.$$

$$\left. \left. - L_{f_2'(y)}\Big|_{y=\tilde{y}} \cdot \frac{\rho_2}{\sqrt{1 - \rho_2^2}} \right) \right\} \delta\tilde{y}$$

Hence, if $f_2(y)$ is optimal then it must satisfy

$$L_{f_2(y)} - \frac{d}{dy}\left(L_{f_2'(y)}\right) = 0 \tag{94}$$

$$L_{f_2'(y)}\Big|_{y=\tilde{y}} + C_B \cdot n \cdot \left(\frac{L}{n} - \tilde{y}\right) = 0 \tag{95}$$

and

$$L\big|_{y=\tilde{y}} - \frac{C_m \cdot n}{V_m} \frac{1}{\sqrt{1 - \rho_2^{\,2}}} - C_B \cdot n\tilde{X} = 0 \tag{96}$$

Using (94) we get

$$\frac{f_2'(y)}{\sqrt{1 + (f_2'(y))^2}} = \alpha(U_o)y + \beta \tag{97}$$

where $\alpha(U_o) = \dfrac{C_B}{\dfrac{C_m}{V_m} + U_o \dfrac{\rho_2}{n}}$ and β is an arbitrary constant.

Combining (97) and (95) we get

$$\alpha(U_o) \cdot \frac{L}{n} + \beta = \frac{\alpha(U_o) \cdot U_o}{C_B \cdot n} \tag{98}$$

Combining (94), (95), and (96) we get

$$\alpha(U_o) \cdot \tilde{y} + \beta = \rho_2 \tag{99}$$

Further, we have by (93) and (97)

$$X_1 + \frac{1}{\alpha}\left\{\sqrt{1 - (\alpha y_1 + \beta)^2} - \sqrt{1 - (\alpha\tilde{y} + \beta)^2}\right\}$$

$$= X_f^{(1)} + \frac{\rho_2}{\alpha}\left\{\sin^{-1}(\alpha\tilde{y} + \beta) - \sin^{-1}(\alpha y_1 + \beta)\right\} \tag{100}$$

Equations (98), (99), and (100) have three unknowns, i.e., α, β, and $\tilde{y}$ and hence we can solve for these quantities and the optimal firebreak path is given by

$$X_m = f_2^*(y)$$

where

$$f_2^*(y) = \begin{cases} X_1 + \frac{1}{\alpha}\left\{\sqrt{1-(\alpha y_1+\beta)^2} - \sqrt{1-(\alpha y+\beta)^2}\right\} & \text{for } y_1 \leq y \leq \tilde{y} \\ X_1 + \frac{1}{\alpha}\left\{\sqrt{1-(\alpha y_1+\beta)^2} - \sqrt{1-\rho_2^{\,2}}\right\} + \frac{\rho_2}{\sqrt{1-\rho_2^{\,2}}} \cdot y & \text{for } \tilde{y} \leq y \leq \frac{L}{n} \end{cases}$$

(101)

The first part of the equation, i.e., for $y_1 \leq y \leq \tilde{y}$, may be rewritten as

$$\text{(102)} \qquad \left\{X_m - \left(X_1 + \frac{1}{\alpha}\sqrt{1-(\alpha y_1+\beta)^2}\right)\right\}^2 + \left\{y + \beta/\alpha\right\}^2 = \frac{1}{\alpha^2}$$

Thus, this part of the curve is a circular arc with center $\left\{X_1 + \frac{1}{\alpha}\sqrt{1-(\alpha y_1+\beta)^2},\ -\beta/\alpha\right\}$ and radius $1/\alpha$. The latter part is a straight line tangent to the circle at $(f_2(\tilde{y}), \tilde{y})$.

Substituting the equation of the optimal firebreak path in (89) we can find the optimal cost $K_2(n;\ N_1, N_2, X_1, y_1, X_f^{(1)} | V_f^{(1)}, V_f^{(2)})$ for any arbitrary values of N_1, N_2, $X_f^{(1)}$, $V_f^{(2)}$, y_1 and $X_1 \geq X_f^{(1)}$. Let this cost be given by

$$K_2(n;\, N_1,\, N_2,\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)},\, V_f^{(2)})$$

$$= C_F + C_S(N_2 - N_1) + \frac{C_m \cdot n}{V_m}\int_{y_1}^{L/n} \sqrt{1 + (f_2^{*\prime}(y))^2} \cdot dy$$

$$+ C_B \cdot n \int_{y_1}^{L/n} f_2^*(y)dy$$

$$\triangleq C_S(N_2 - N_1) + G_2(n;\, N_2,\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)},\, V_f^{(2)})$$

(103)

Definition:

$$K_2(n;\, N_1,\, N_2,\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)})$$

$$= E_{V_f^{(2)}}[K_2(n;\, N_1,\, N_2,\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)},\, V_f^{(2)})]$$

$$= \int_0^{\hat{V}_f} K_2(n;\, N_1,\, N_2,\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)},\, X)d\Phi_2(X)$$

where

$$\Phi_2(X) = \Pr\{V_f^{(2)} \leq X|V_f^{(1)}\}$$

It can be shown, in a manner similar to that of proposition 2, that $K_2(n;\, N_1,\, N_2,\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)})$ is $\delta_2(n;\, N_2^{\,o},\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)})$-convex in N_2 for $N_2 \geq N_2^{\,o}$, where

$$\delta_2(n;\, N_2{}^{o},\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)} = G_2(n;\, N_2{}^{o},\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)})$$

$$= E_{V_f^{(2)}}\left\{G_2(n;\, N_2{}^{o},\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)},\, V_f^{(2)}\right\}$$

$$= \int_o^{\hat{V}_f} G_2(n;\, N_2{}^{o},\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)},\, X)d\Phi(X)$$

But since we know $N_2^* \geq N_1$ and have imposed the condition that $N_2^* \geq \hat{N}$, we need enumerate only over the interval $\left[\bar{N},\, \bar{N} + \frac{\delta_2(n;\, \bar{N},\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)})}{C_S}\right]$ where $\bar{N} = \text{Max}(N_1, \hat{N})$, the reason for this being similar to those in the one-period case, i.e., for

$$N \geq \bar{N} + \frac{\delta_2(n;\, \bar{N},\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)})}{C_S},$$

$$K_2(n;\, N_1,\, N,\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)}) \geq K_2(n;\, N_1,\, \bar{N},\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)})$$

For both cases, i.e., $X_f^{(1)} = X_1$ and $X_f^{(1)} < X_1$, $K_2(n;\, N_1,\, N_2^*,\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)})$ is a nonincreasing function of N_1. For this let us consider (103) with $N_2 = N_2^*$

$$K_2(n;\, N_1,\, N_2,\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)} = C_S(N_2^* - N_1) + G_2(n;\, N_2^*,\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)})$$

For $N_1 < N_2^*$, $G_2(n;\, N_2^*,\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)})$ does not depend upon N_1 and

$$\frac{\partial K_2(n;\, N_1,\, N_2^*,\, X_1,\, y_1,\, X_f^{(1)}|V_f^{(1)})}{\partial N_1} = -\, C_S < 0\ .$$

But $N_2^* \geqq N_1$. Hence the only other case is $N_1 = N_2^*$. But for this

$$K_2(n;\ N_1,\ N_2^*,\ X_1,\ y_1,\ X_f^{(1)}|V_f^{(1)}) = G_2(n;\ (N_2^* = N_1),\ X_1,\ y_1,\ X_f^{(1)}|V_f^{(1)})$$

But we know that $G_2(n;\ N_2,\ X_1,\ y_1,\ X_f^{(1)}|V_f^{(1)})$ is a nonincreasing function of N_2 and hence so is $K_2(n;\ N_1,\ N_2^*,\ X_1,\ y_1,\ X_f^{(1)}|V_f^{(1)})$ in N_1. Thus for $N_2^* \geqq N_1$, $K_2(n;\ N_1,\ N_2^*,\ X_1,\ y_1,\ X_f^{(1)}|V_f^{(1)})$ is a nonincreasing function of N_1.

Now we shall consider the variational problem for the first period.

(b) <u>Variational Problem for Period I.</u>

The variational problem for Period I can be stated as follows:

$$\underset{f_1(y)}{\text{Min}} \left\{K_1(n;\ N_1,\ f_1(y)) + K_2(n;\ N_1,\ N_2^*,\ X_1,\ y_1,\ X_f^{(1)}|V_f^{(1)})\right\}$$

(104)

Subject to

(105) $$f_1(y) \geqq \rho_1 \int_0^y \sqrt{1 + (f_1'(y))^2} \cdot dy \qquad y \in [0,\ y_1]$$

(106) $$\frac{1}{V_m(N_1/n)} \int_0^{y_1} \sqrt{1 + (f_1'(y))^2} \cdot dy = T$$

where

(107) $$K_1(n;\ N_1,\ f_1(y)) = C_F + C_S \cdot N_1 + C_m \cdot N_1 \cdot T + C_B \cdot n \cdot \int_0^{y_1} f_1(y) dy$$

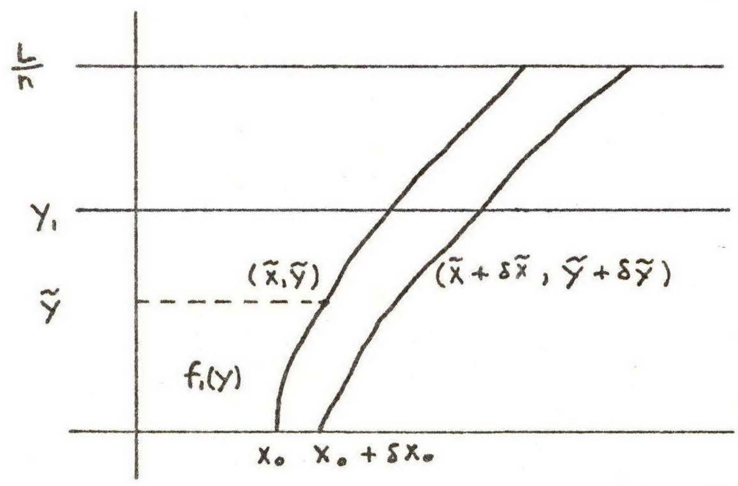
Note that $K_2(n;\ N_1,\ N_2^*,\ X_1,\ y_1,\ X_f^{(1)}|V_f^{(1)})$ is dependent on $f_1(y)$ via the parameters $(X_1,\ y_1)$ and that the optimization on $(X_1,\ y_1)$ can be done directly by applying the principles of variational calculus to take

these also into account. Equation (106) can be modified for convenience into the following form

(108) $$\rho_1 \int_0^{y_1} \sqrt{1 + (f_1'(y))^2} \cdot dy = V_f^{(1)} \cdot T$$

Here we distinguish two cases: (i) when men and fire meet for some $y = \tilde{y} < y_1$, and (ii) when they do not meet for any $y < y_1$. For these two cases, we shall make use of the appropriate cost functions for the second period. Note that the second case takes account of the possibility that the men and the fire do not meet at all in the first period. We shall treat the two cases separately.

<u>Case</u> <u>(i)</u> <u>$\tilde{y} < y_1$</u>



Firebreak variations for the first period.

For this case, we can replace condition (105) by (109) as in Chapter II:

$$(109) \qquad \int_0^{\tilde{y}} \rho_1 \sqrt{1 + (f_1'(y))^2} \cdot dy = f_1(\tilde{y})$$

and condition (108) can be rewritten as

$$\rho_1 \int_0^{\tilde{y}} \sqrt{1 + (f_1'(y))^2} \cdot dy = V_f^{(1)}\, T - \frac{y_1 - \tilde{y}}{\sqrt{\left(V_m \frac{N_1}{n}\right)^2 - \left(V_f^{(1)}\right)^2}}$$

(110)

Define

$$J(f_1(y)) = \int_0^{\tilde{y}} \left\{ \mathcal{L}(y,\, f_1(y),\, f_1'(y)) - \frac{U_o f_1(0)}{\tilde{y}} - \frac{U_o' V_f^{(1)} \cdot T}{\tilde{y}} \right\} dy$$

$$+ C_B \cdot n \left\{ \tilde{X}(y_1 - \tilde{y}) + \frac{1}{2}(y_1 - \tilde{y})^2 \frac{\rho_1}{\sqrt{1 - \rho_1^2}} \right\}$$

$$+ K_2(n;\, N_1,\, N_2^*;\, X_1 = X_f^{(1)},\, y_1(\tilde{X}, \tilde{y}),\, X_f^{(1)} | V_f^{(1)})$$

$$+ \frac{U_o'(y_1 - \tilde{y})}{\sqrt{\left(V_m \frac{N_1}{n}\right)^2 - \left(V_f^{(1)}\right)^2}}$$

where

$$\mathcal{L}(y,\, f_1(y),\, f_1'(y) = [C_B \cdot n\, f_1(y) + (U_o + U_o')\, \rho_1 \sqrt{1 + (f_1'(y))^2} - U_o f_1'(y)]$$

[Note that $K_2(\cdot)$ is that of case (i) in the analysis for the second period.]

We shall consider variations of the type $f_1(y) + h_1(y)$ where $h_1(y)$ is continuous and differentiable and satisfies

$$h(0) = \delta X_o$$

$$\delta\tilde{X} - \frac{\rho_1}{\sqrt{1 - \rho_1^2}} \delta\tilde{y} = h(\tilde{y})$$

$$\Delta J(f_1(y)) = \int_o^{\tilde{y}+\delta\tilde{y}} \left\{ \mathcal{L}(y, f_1(y) + h_1(y), f_1'(y) + h_1'(y)) - \frac{U_o[f_1(0) + \delta X_o]}{\tilde{y} + \delta\tilde{y}} - \frac{U_o' V_f^{(1)} T}{\tilde{y} + \delta\tilde{y}} \right\} dy$$

$$+ C_B n \left[(\tilde{X}+\delta\tilde{X})(y_1(\tilde{X}+\delta\tilde{X}, \tilde{y}+\delta\tilde{y}) - \tilde{y}-\delta\tilde{y}) + \frac{1}{2} (y_1(\tilde{X}+\delta\tilde{X}, \tilde{y}+\delta\tilde{y}) - \tilde{y} - \delta\tilde{y})^2 \frac{\rho_1}{\sqrt{1 - \rho_1^2}} \right]$$

$$+ K_2(n; N_1, N_2^*; X_1 = X_f^{(1)}, y_1(\tilde{X} + \delta\tilde{X}, \tilde{y} + \delta\tilde{y}), X_f^{(1)} | V_f^{(1)})$$

$$+ \frac{U_o'[y_1(\tilde{X} + \delta\tilde{X}, \tilde{y} + \delta\tilde{y}) - \tilde{y} - \delta\tilde{y}]}{\sqrt{\left(\frac{V_m N_1}{n}\right)^2 - \left(V_f^{(1)}\right)^2}}$$

$$- \int_o^{\tilde{y}} \left(\mathcal{L}(y, f_1(y), f_1'(y)) - \frac{U_o f_1(0)}{\tilde{y}} - \frac{U_o' V_f^{(1)} \cdot T}{\tilde{y}} \right) dy$$

$$- C_B n \left[\tilde{X}(y_1(\tilde{X}, \tilde{y}) - \tilde{y}) + \frac{1}{2} (y_1(\tilde{X}, \tilde{y}) - \tilde{y})^2 \frac{\rho_1}{\sqrt{1 - \rho_1^2}} \right]$$

$$- K_2[n; N_1, N_2^*, X_1 = X_f^{(1)}, y_1(\tilde{X}, \tilde{y}), X_f^{(1)} | V_f^{(1)}]$$

$$- \frac{U_o'[y_1(\tilde{X}, \tilde{y}) - \tilde{y}]}{\sqrt{\left(\frac{V_m N_1}{n}\right)^2 - \left(V_f^{(1)}\right)^2}}$$

Using the fact that

$$y_1 = \tilde{y} + \frac{(X_f^{(1)} - \tilde{X})}{V_f^{(1)}} \cdot \sqrt{\left(\frac{V_m N_1}{n}\right)^2 - \left(V_f^{(1)}\right)^2}$$

$$= \tilde{y} + \frac{(X_f^{(1)} - \tilde{X})}{\rho_1} \sqrt{1 - \rho_1^{\,2}}$$

$$\Delta J(f_1(y)) = \int_0^{\tilde{y}+\delta\tilde{y}} \left\{ \mathcal{L}(y, f_1(y) + h_1(y), f_1'(y) + h_1'(y)) - \frac{U_o[f_1(0) + \delta X_o]}{\tilde{y} - \delta\tilde{y}} - \frac{U_o' V_f^{(1)} \cdot T}{\tilde{y} + \delta\tilde{y}} \right\} dy$$

$$- \int_0^{\tilde{y}} \left[\mathcal{L}(y, f_1(y), f_1'(y)) - \frac{U_o[f_1(0)]}{\tilde{y}} - \frac{U_o' V_f^{(1)} \cdot T}{\tilde{y}} \right] dy$$

$$+ K_2[n; N_1, N_2^*, X_1 = X_f^{(1)}, y_1(\tilde{X} + \delta\tilde{X}, \tilde{y} + \delta\tilde{y}), X_f^{(1)} | V_f^{(1)}]$$

$$- K_2[n, N_1, N_2^*, X_1 = X_f^{(1)}, y_1(\tilde{X}, \tilde{y}), X_f^{(1)} | V_f^{(1)}]$$

$$+ \frac{U_o' \cdot (X_f^{(1)} - \tilde{X} - \delta\tilde{X})}{V_f^{(1)}} - U_o' \left(\frac{X_f^{(1)} - \tilde{X}}{V_f^{(1)}} \right)$$

$$+ C_B \cdot n \left[(\tilde{X} + \delta\tilde{X}) \frac{(X_f^{(1)} - \tilde{X} - \delta\tilde{X})}{\rho_1} \sqrt{1 - \rho_1^{\,2}} + \frac{1}{2} \frac{(X_f^{(1)} - \tilde{X} - \delta\tilde{X})^2}{\rho_1} \sqrt{1 - \rho_1^{\,2}} \right]$$

$$- C_B \cdot n \left[\tilde{X} \frac{(X_f^{(1)} - \tilde{X})}{\rho_1} \sqrt{1 - \rho_1^{\,2}} + \frac{1}{2} \frac{(X_f^{(1)} - \tilde{X})^2}{\rho_1} \sqrt{1 - \rho_1^{\,2}} \right]$$

and

$$\delta J = \int_{o}^{\tilde{y}} \left\{ \mathcal{L}_{f_1(y)} - \frac{d}{dy}\left(\mathcal{L}_{f_1'(y)}\right) \right\} h_1(y)dy$$

$$- \left[\mathcal{L}_{f_1'(y)}\Big|_{y=o} + U_o \right] \delta X_o$$

$$+ \left[C_B \cdot n \frac{\sqrt{1 - \rho_1^{\,2}}}{\rho_1} (- \tilde{X}) + \mathcal{L}_{f_1'(y)}\Big|_{y=\tilde{y}} + \frac{\partial K_2}{\partial \tilde{X}} \right] \delta \tilde{X}$$

$$+ \left[\mathcal{L}\Big|_{y=\tilde{y}} + \frac{\partial K_2}{\partial \tilde{y}} \right] \delta \tilde{y}$$

where δX_o, $h_1(y)$, $\delta \tilde{X}$, and $\delta \tilde{y}$ are arbitrary. Thus if $f_1(y)$ is optimal then it satisfies (111) - (114).

(111) $$\mathcal{L}_{f_1(y)} - \frac{d}{dy}\left(\mathcal{L}_{f_1'(y)}\right) = 0$$

(112) $$\mathcal{L}_{f_1'(y)}\Big|_{y=o} + U_o = 0$$

(113) $$\mathcal{L}_{f_1'(y)}\Big|_{y=\tilde{y}} - C_B \cdot n \cdot \frac{\sqrt{1 - \rho_1^{\,2}}}{\rho_1} + \frac{\partial K_2}{\partial \tilde{X}} = 0$$

(114) $$\mathcal{L}\Big|_{y=\tilde{y}} + \frac{\partial K_2}{\partial \tilde{y}} = 0$$

Using (111) we get

$$C_B \cdot n - (U_o + U_o')\rho_1 \frac{d}{dy} \left\{ \frac{f_1'(y)}{\sqrt{1 + (f_1'(y))^2}} \right\} = 0$$

$$(115) \qquad \therefore \quad \frac{f_1'(y)}{\sqrt{1 + (f_1'(y))^2}} = \left(\frac{C_B}{(U_o + U_o') \frac{\rho_1}{n}} \right) \cdot y + \beta$$

(115) and (112) implies $\beta = 0$.

$$(116) \qquad \therefore \; f_1^*(y) = X_o + \frac{1}{\alpha} \left\{ 1 - \sqrt{1 - (\alpha y)^2} \right\} \qquad \text{for } 0 \leqq y \leqq \tilde{y}$$

where

$$\alpha = \frac{C_B}{(U_o + U_o') \frac{\rho_1}{n}}$$

Thus, as before, the optimal firebreak path is a circle up to $y = \tilde{y}$ and then it is a straight line tangent to this circle at $(\tilde{X}, \tilde{y})$. The circle has a center at $(X_o + 1/\alpha, 0)$ and a radius $= 1/\alpha$.

Now we are left with three unknowns $\tilde{y}$, X_o, and α. We can solve for these using (113), (114), and substituting (116) in (110). Thus we can solve the variational problem for this case and let us denote the optimal path by $f_1^*(y)$ where

$$f_1^*(y) = \begin{cases} X_o + 1/\alpha \left\{ 1 - \sqrt{1 - (\alpha y)^2} \right\} & \text{for } 0 \leqq y \leqq \tilde{y} \\ X_o + 1/\alpha \left\{ 1 - \sqrt{1 - (\alpha \tilde{y})^2} \right\} + \dfrac{\rho_1}{\sqrt{1 - \rho_1^{\,2}}} (y - \tilde{y}) & \text{for } \tilde{y} \leqq y \leqq y_1 \end{cases}$$

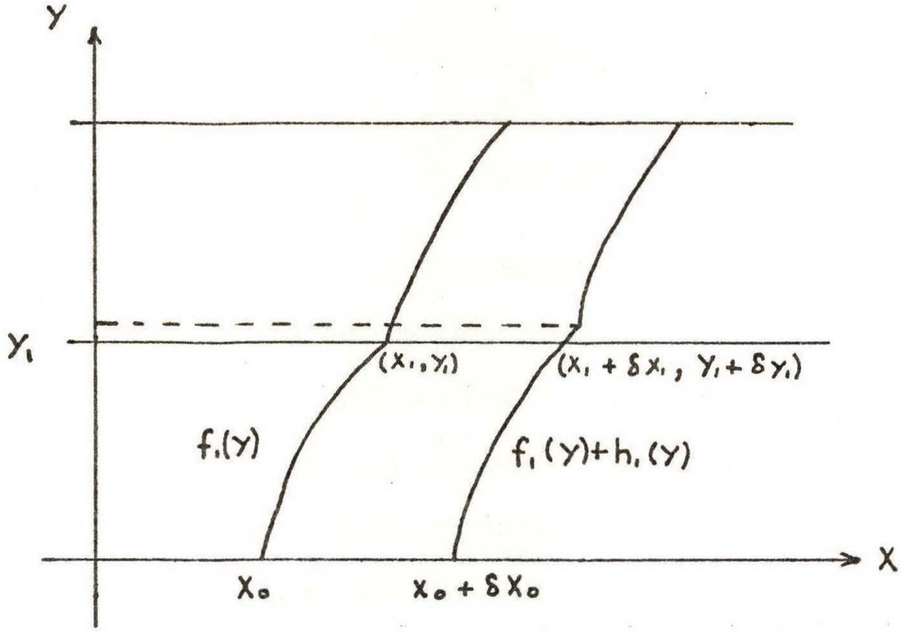
Let us now consider case (ii) when the fire and the men do not meet for any $y < y_1$.

Case (ii) $\tilde{y} \nless y_1$

For this case, we can replace condition (105) by

$$\int_0^{y_1} \rho_1 \sqrt{1 + (f_1'(y))^2}\, dy \leq f_1(y_1) \tag{117}$$

and the rest of the problem is unchanged.



Firebreak variations for Period I.

Define

$$J(f_1(y)) = \int_0^{y_1} \left\{ \mathcal{L}(y, f_1(y), f_1'(y)) - \frac{U_o f_1(0)}{y_1} - \frac{U_o' V_f^{(1)} \cdot T}{y_1} \right\} dy$$

$$+ K_2(n;\, N_1,\, N_2^*,\, X_1,\, y_1,\, X_f^{(1)} | V_f^{(1)})$$

Note that the cost function $K_2(\cdot)$ used here is that of the case when $X_1 \geq X_f^{(1)}$.

$$\Delta J(f_1(y)) = \int_0^{y_1+\delta y_1}\Bigg\{\mathcal{L}(y, f_1(y) + h_1(y), f_1'(y) + h_1'(y)) - \frac{U_o(f_1(0) + \delta X_o}{y_1 + \delta y_1} - \frac{U_o'V_f^{(1)} \cdot T}{y_1 + \delta y_1}\Bigg\}\, dy$$

$$- \int_0^{y_1}\left\{\mathcal{L}(y, f_1(y), f_1'(y)) - \frac{U_o f_1(0)}{y_1} - \frac{U_o'V_f^{(1)} \cdot T}{y_1}\right\} dy$$

$$+ K_2(n;\, N_1,\, N_2^*,\, X_1 + \delta X_1,\, y_1 + \delta y_1,\, X_f^{(1)} | V_f^{(1)})$$

$$- K_2(n;\, N_1,\, N_2^*,\, X_1,\, y_1,\, X_f^{(1)} | V_f^{(1)})$$

and

$$\delta J = \int_0^{y_1}\left\{\mathcal{L}_{f_1(y)} - \frac{d}{dy}\left(\mathcal{L}_{f_1'(y)}\right)\right\} h_1(y) + \left[\mathcal{L}_{f_1'(y)}\Big|_{y=y_1} + \frac{\partial K_2}{\partial X_1}\right] \delta X_1$$

$$- \left[\mathcal{L}_{f_1'(y)}\Big|_{y=o} + U_o\right] \delta X_o + \left[\mathcal{L}\Big|_{y=y_1} + \mathcal{L}_{f_1'(y)}\Big|_{y=y_1} + \frac{\partial K_2}{\partial y_1}\right] \delta y_1$$

If $f_1(y)$ is optimal, then since $h_1(y)$, δX_1, δy_1, and δX_o are arbitrary, we have

(118) $$\mathcal{L}_{f_1(y)} - \frac{d}{dy}\left(\mathcal{L}_{f_1'(y)}\right) = 0$$

(119) $$\mathcal{L}_{f_1'(y)}\Big|_{y=o} + U_o = 0$$

(120) $$\mathcal{L}_{f_1'(y)}\Big|_{y=y_1} + \frac{\partial K_2}{\partial X_1} = 0$$

$$(121) \qquad \mathcal{L}\Big|_{y=y_1} + \mathcal{L}_{f_1'(y)}\Big|_{y=y_1} + \frac{\partial K_2}{\partial y_1} = 0$$

Using equation (118) we get

$$(122) \qquad \frac{f_1'(y)}{\sqrt{1 + (f_1'(y))^2}} = \frac{C_B}{(U_o + U_o')\,\frac{\rho_1}{n}} \cdot y + \beta$$

and combining (119) and (122) we get $\beta = 0$. Thus, once again, the optimal path is an arc of a circle with center $(X_o + 1/\alpha,\ 0)$ and radius $1/\alpha$. Again we have three unknowns X_o, y_1, and $(U_o + U_o')$, and we can solve for these using equations (120), (121), and (108) and obtain $f_1^*(y)$ and hence the total cost, where $f_1^*(y)$ is given by

$$f_1^*(y) = X_o + \frac{1}{\alpha}\left\{1 - \sqrt{1 - (\alpha y)^2}\right\} \quad \text{for } 0 \leq y \leq y_1$$

where

$$\alpha = \frac{C_B}{(U_o + U_o')\,\frac{\rho_1}{n}} .$$

In all these cases let us denote the suboptimized cost function by $K_1(n;\ N_1|V_f^{(1)})$ where

$$\begin{aligned} K_1(n;N_1|V_f^{(1)}) &= C_F + C_S \cdot N_1 + \frac{C_m \cdot n}{V_m}\int_0^{y_1^*} \sqrt{1 + (f_1^{*\prime}(y))^2} \cdot dy \\ &\quad + C_B n \int_0^{y_1^*} f_1^*(y)dy\ K_2(n;\ N_1,\ N_2^*,\ X_1^*,\ y_1^*,\ X_f^{(1)}|V_f^{(1)}) \\ &\triangleq C_S \cdot N_1 + G_1(n;\ N_1|V_f^{(1)}) \end{aligned}$$

By an argument similar to proposition 1 of this chapter we can show $G_1(n;\ N_1|V_f^{(1)})$ is a nonincreasing function of N_1 and hence $K_1(n;\ N_1|V_f^{(1)})$ is $\delta(n;\ N_1{}^o|V_f^{(1)})$-convex in N_1 for $N_1 \geq N_1{}^o$ and hence so is $K_1(n;\ N_1)$ where

$$K_1(n;\ N_1) = E_{V_f^{(1)}} \left\{ K_1(n;\ N_1 | V_f^{(1)} \right\}$$

$$= \int_0^{\hat{V}_f} K_1(n;\ N_1 | X) d\Phi_1(X)$$

where $\Phi_1(X) = \Pr \left\{ V_f^{(1)} \leq X \right\}$.

Thus we can determine N_1^* by using the procedure outlined in the one-period case, where N_1^* yields the global minimum of $K_1(n;\ N_1)$.

Now we shall extend this to the case of more than two periods.

In the above analysis, we have to compute $K_2(n;\ N_1,\ N_2^*,\ X_1,\ y_1,\ X_f^{(1)} | V_f^{(1)})$ for various values of X_1, y_1, N_1 and $V_f^{(1)}$ since N_2^* depends implicitly on X_1, y_1, etc., and no closed form solution is available. Although these computations are difficult, using the properties of δ-convexity, the computations may be feasible on a large electronic computer. Since the problem is one of dynamic programming in 3-state variables, i.e. $(N_i,\ X_i,\ y_i)$, it is obvious that a large computer is essential. The purpose of this discussion is to show that such computations are feasible.

5. Multiperiod Analysis

Let us consider a p-period problem. Then for the variational problems of all periods except the first, the initial point is fixed and a condition of the type (108) is required for all periods except the last where the firebreak runs up to $y = L/n$ regardless of the time taken. We start by solving for $p = 1$, then for $p = 2$, and so on. Finally, we give a stopping rule whereby we can stop the analysis for some value of p, say $p = P$, and then consider the optimized costs of p-period analyses for $p = 1, \ldots \underline{P}$ to determine the optimal number of periods p^*. We give below an example of the variational problems encountered here.

Let

$$K_i^{(p)}(n;\, N_1,\, \ldots,\, N_i,\, X_{i-1},\, y_{i-1},\, X_f^{(i-1)}, f_i(y)\,|\,V_f^{(1)},\, \ldots,\, V_f^{(i)})$$

$$= C_F + C_S(N_i - N_{i-1}) + \frac{C_m n}{V_m}\int_{y_{i-1}}^{y_i} \sqrt{1 + (f_i'(y))^2}\cdot dy + C_B\cdot n\int_{y_{i-1}}^{y_i} f_i(y)dy$$

$$+ K_{i+1}^{(p)}(n;\, N_1,\, \ldots,\, N_i,\, N_{i+1}^*,\, X_i,\, y_i\,|\,V_f^{(1)},\, \ldots,\, V_f^{(i)})$$

(124)

where $K_{i+1}^{(p)}(n;\, N_1,\, \ldots,\, N_i,\, N_{i+1}^*,\, X_i,\, y_i\,|\,V_f^{(1)},\, \ldots,\, V_f^{(i)})$ is the optimized cost over the rest of the periods and is known when solving the ith period problem. Then a typical variational problem would be

$$\min_{f_i(y)} K_i^{(p)}(n;\, N_1,\, \ldots,\, N_i,\, X_{i-1},\, y_{i-1},\, X_f^{(i-1)}\,|\,V_f^{(1)},\, \ldots,\, V_f^{(i)})$$

Subject to

(125) $$f_i(y) \geq \left\{\rho_i\int_{y_{i-1}}^{y} \sqrt{1 + (f_i'(y))^2}\cdot dy\right\} + X_f^{(i-1)}$$

(126) $$\rho_i\int_{y_{i-1}}^{y_i} \sqrt{1 + (f_i'(y))^2}\cdot dy = V_f^{(i)}\cdot T$$

(127) $$f_i(y_{i-1}) = X_{i-1}$$

with $y_p = L/n$, $y_o = 0$

Condition (127) does not apply when $i = 1$ and (126) does not apply for $i = p$.

The analyses of the variational problems are similar to the ones in the two-period case and so we do not go into the details here. But we shall describe certain properties regarding the suboptimized cost functions obtained by using the optimal paths for arbitrary crew size. Let the total cost from period i onwards be denoted by

$$K_i^{(p)} (n; N_1, \ldots, N_i | V_f^{(1)}, \ldots, V_f^{(i)})$$

$$= K_i^{(p)} (n; N_1, \ldots, N_i | V_f^{(1)}, \ldots, V_f^{(i)})$$

$$+ K_{i+1}^{(p)} (n; N_1, \ldots, N_i, N_{i+1}^* | V_f^{(1)}, \ldots, V_f^{(i)})$$

$$\triangleq C_S(N_i - N_{i-1}) + G_i^{(p)} (n; N_1, \ldots, N_i | V_f^{(1)}, \ldots, V_f^{(i)})$$

(122a)

The superscript p denotes that we are having a p-period analysis.

<u>Proposition 4</u>

$K_i^{(p)} (n; N_1, \ldots, N_i | V_f^{(1)}, \ldots, V_f^{(i)})$ is $\delta_i^{(p)} (n; N_1, \ldots, N_{i-1}, N_i^{\circ} | V_f^{(1)}, \ldots, V_f^{(i)})$-convex in N_i for $N_i \geq N_i^{\circ}$ where

$$\delta_i^{(p)}(n; N_1, \ldots, N_{i-1}, N_i^{\circ} | V_f^{(1)}, \ldots, V_f^{(i)})$$

$$= G_i^{(p)}(n; N_1, \ldots, N_{i-1}, N_i^{\circ} | V_f^{(1)}, \ldots, V_f^{(i)})$$

Proof is similar to that of proposition 2.

Proposition 5

Let $\delta_i^{(p)}(n; N_1, \ldots, N_{i-1}, N_i | V_f^{(1)}, \ldots, V_f^{(i-1)})$

$$= E_{V_f^{(i)}}\left\{\delta_i^{(p)}(n; N_1, \ldots, N_{i-1}, N_i^* | V_f^{(1)}, \ldots, V_f^{(i)}\right\}$$

$$= \int_o^{\hat{V}_f} G_i^{(p)}(n; N_1, \ldots, N_{i-1}, N_i^* | V_f^{(1)}, \ldots, V_f^{(i-1)}, X)d\Phi_i(X)$$

where $\Phi_i(X) = \Pr\left\{V_f^{(i)} \leq X | V_f^{(1)}, \ldots, V_f^{(i-1)}\right\}$.

Then

$\delta_i^{(p)}(n, N_1, \ldots, N_i | V_f^{(1)}, \ldots, V_f^{(i-1)})$

$$\geq \delta_{i+1}^{(p)}(n; N_1, \ldots, N_i, N_{i+1}^* | V_f^{(1)}, \ldots, V_f^{(i-1)})$$

where

$\delta_{i+1}^{(p)}(n; N_1, \ldots, N_i, N_{i+1}^* | V_f^{(1)}, \ldots, V_f^{(i-1)})$

$$= E_{V_f^{(i)}}(\delta_{i+1}^{(p)}(n; N_1, \ldots, N_i, N_{i+1}^* | V_f^{(1)}, \ldots, V_f^{(i)}))$$

Proof: Since $K_{i+1}^{(p)}(n; N_1, \ldots, N_i, N_{i+1}^* | V_f^{(1)}, \ldots, V_f^{(i)})$ is $\delta_{i+1}^{(p)}(n; N_1, \ldots, N_i, N_{i+1}^*)$-convex and

$K_i^{(p)}(n; N_1, \ldots, N_i | V_f^{(1)}, \ldots, V_f^{(i)})$

$$= K_{i+1}^{(p)}(n; N_1, \ldots, N_i, N_{i+1}^* | V_f^{(1)}, \ldots, V_f^{(i)}) + K_i^{(p)}(\cdot)$$

$\therefore \delta_i^{(p)}(n; N_1, \ldots, N_i | V_f^{(1)}, \ldots, V_f^{(i)} \geq \delta_{i+1}^{(p)}(n; N_1, N_2, \ldots, N_i, N_{i+1}^* |$
$V_f^{(1)}, \ldots, V_f^{(i)}$ from which the proposition follows.

These two results can lessen the computation tremendously.

6. Determination of Optimal Number of Periods

Proposition 6

Let us suppose that we have solved the p-period problem and find that (123) holds with $V_f^{(p)} = \hat{V}_f$, the highest possible value for the realized value of the fire velocity.

$$(123) \qquad \left[\frac{1}{V_m(N_p^*/n)} \int_{y_{p-1}}^{L/n} \sqrt{1 + (f_p^{*\prime}(y))^2} \cdot dy\right] \leq T$$

then $K_1^{(p)} \leq K_1^{(j)}$ for all $j \geq p$

where $K_1^{(p)}$ is the optimized total expected cost for the p-period problem. This is evident from the fact that $N_1^*, \ldots, N_p^*$ are the optimal crew sizes and hence if we consider a (p + 1)-period problem, then whatever be the values of $N_1, \ldots, N_{p+1}$, the total expected cost over the first p-periods is greater or equal to $K^{(p)}$ and hence the above result is true. Thus the optimal number of periods p^* is given by

$$K^{(p^*)} = \text{Min}\ [K^{(1)}, \ldots, K^{(p)}]$$

where p is the minimum value for which (123) holds.

7. Numerical Algorithm for Stochastic Case

Step I: Set p = 1.

Step II: Solve the p-period problem.

(1) Set q = p

(2) Solve the qth period's problem: i.e.

(a) For arbitrary n, N_1, ..., N_q, y_q, $X_f^{(q)}$, $V_f^{(1)}$, ..., $V_f^{(q)}$ and $X_q \geqq X_f^{(q)}$ solve the variational problem for the qth period to give $K_q^{(p)}(n; N_1, \ldots, N_q, X_q, y_q, X_f^{(q)} | V_f^{(1)}, \ldots, V_f^{(q)})$

(b) Optimize $K_q^{(p)}(n; N_1, \ldots, N_q, X_q, y_q, X_f^{(q)} | V_f^{(1)}, \ldots, V_f^{(q-1)})$ over N_q to get N_q^* and hence $K_q^{(p)}(n; N_1, \ldots, N_{q-1}, N_q^*, X_q, y_q, X_f^{(q)} | V_f^{(1)}, \ldots, V_f^{(q-1)})$ and go to (c)

(c) If q - 1 = 0, go to (III) and if q - 1 > 0 then set q = q - 1 and go to 2(a)

Step III: Check if equation (123) holds. If yes then go to IV. If no then set p = p + 1 and go to II.

Step IV: Find p^* which satisfies $K^{(p^*)} = \text{Min}\ [K_1^{(1)}, \ldots, K_1^{(p)}]$ and stop. Current solution is optimal.

8. Conclusions and Remarks

The values of p* and N_1^* are calculated once and for all to minimize the total expected cost of controlling the fire. These quantities depend upon the cost parameters, the probability distribution and hence $\hat{V}_f$ and the review time T. But for a given forest the cost parameters are known and if the meteorological conditions of a particular forest are sufficiently well studied the frequency distribution is also known. Also, the review time T is fixed. Therefore, for any particular forest, we can calculate p* and N_1^* beforehand and once these numbers are calculated the rest of the work is considerably simpler. Although this calculation takes

a large amount of time, it is feasible to use this analysis in practice since it is done only once and beforehand. In any case, it is p* and N_1^* that are required for initial planning and the other quantities can be calculated with much less work since X_1, y_1, $X_f^{(1)}$, etc., would then be known. Since p* and N_1^* are determined on the basis of the distribution of V_f and $\hat{V}_f$ amongst other things, if one tends to be cautious and over-estimates $\hat{V}_f$ in trying to be safe, then the values of p* and N_1^* would tend to be larger. This would mean a larger expected cost due to a larger value of N_1^* and more computational effort since p* is larger. Thus it may be worthwhile to estimate $\hat{V}_f$ carefully.

Since the values of p* and N_1^* are computed beforehand for any forest, they can be used as soon as a fire is reported in the forest. In the beginning, the realized value of the fire velocity in the first period is used to determine the actual firebreak path. Also it determines the planned assignment of men at the beginning of the second period. These men are employed in the firebreak path of the second period, depending upon the realized value of the velocity of the fire at the beginning of the second period. This same realized value determines the number of men put into work at the beginning of the third period. Thus the allocation of men to the fire and the actual path for the firebreak depend upon the sequences of the realized values of the fire velocity at the beginning of each period. This procedure is continued until the fire is enclosed by the firebreak.

APPENDIX

In this section some properties of quasi-convex functions that have been used in Chapter II are studied.

Definition:

A function f(X) is quasi-convex iff

$$f(\lambda X^{(1)} + \overline{1 - \lambda} X^{(2)}) \leq \max[f(X^{(1)}), f(X^{(2)})]$$

for every pair $X^{(1)}$, $X^{(2)}$ and all $\lambda \in [0, 1]$.

Proposition 1. A twice differentiable function of a single variable X is quasi-convex if

$$(1) \qquad \left\{\frac{df}{dX} = 0\right\} \Rightarrow \left\{\frac{d^2f}{dX^2} > 0\right\}$$

That is, wherever the first derivative vanishes, the second derivative is positive.

Proof: Let us suppose that there exists a function f(X) which satisfies (1) but is not quasi-convex. Then $\exists X^{(1)}, X^{(2)}$ and a $\lambda \in [0, 1] \ni$

$$(2) \qquad f(\lambda X^{(1)} + \overline{1 - \lambda} X^{(2)}) > \max[f(X^{(1)}), f(X^{(2)})]$$

But (2) implies that

$$\max_{X \in [X^{(1)}, X^{(2)}]} f(X) = f(X^{(0)}) > \max[f(X^{(1)}), f(X^{(2)})]$$

and hence $X^{(0)}$ lies in the interior of $[X^{(1)}, X^{(2)}]$ i.e., $X^{(0)} \neq X^{(1)}$ and $X^{(0)} \neq X^{(2)}$. Therefore $X^{(0)}$ must satisfy

(3) $$\left.\frac{df}{dX}\right|_{X=X^{(0)}} = 0$$

and

(4) $$\left.\frac{d^2f}{dX^2}\right|_{X=X^{(0)}} \leq 0$$

But (3) and (4) together contradict (1) which we assumed to be true for f(X). Hence a contradiction ensues.

Proposition 2. An infinitely differentiable function f(X) of a single variable X is quasi-convex if

$$\left\{\frac{df}{dX} = 0\right\} \Rightarrow \begin{cases} \text{The first nonvanishing derivative is} \\ \text{of even order and is positive} \\ \\ \underline{\text{OR}} \text{ all derivatives at this point are} \\ \text{equal to zero.} \end{cases}$$

i.e., wherever the derivative vanishes, it is not a relative maximum with a value higher than its neighboring points.

Proof: Similar to that of proposition 1.

Proposition 3. A scalar function f(X) of a vector X is quasi-convex in X iff $g(\lambda) = f(\lambda X^{(1)} + \overline{1 - \lambda} X^{(2)})$ is quasi-convex in λ for all $X^{(1)}$, $X^{(2)}$ and $\lambda \in [0, 1]$.

One can easily combine propositions (1) and (3) to test whether or not a scalar function of a vector is quasi-convex.

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